PRODUCTION - INVENTORY DECISIONS UNDER ORDER-OCCURRENCE UNCERTAINTY

By

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KEY TO SYMBOLS

A	holding cost associated with an order before disposal in the expected total cost formula, $C(1,1)$
B	holding and disposal cost associated with no order before disposal in the total expected cost formula, $C(1,1)$
C	probability that future production is required if one unit is in stock after production
$C_{aI}(i,m)$	total expected cost component of $C_a(i,m)$ due to realization I
$C_{aII}(i,m)$	total expected cost component of $C_a(i,m)$ due to realization II
$C_a(i,m)$	total expected cost given i units in inventory and m units produced (deterministic case) or m units in inventory after production (stochastic case) when production is required and the optimal disposal of all remaining units at one point policy is followed
$C_a(i,m,S)$	total expected cost given i units in inventory and m units produced (deterministic case) or m units in inventory after production (stochastic case) when production is required and disposal times are specified for the bulk disposal policy
cbest	least-cost solution known
c_d	linear cost of disposal per unit
$c_d(1)$	cost of disposal per unit
$C_d(i,m)$	the disposal cost component of $C(i,m)$
$\underline{C}_d(i,m)$	underestimate of $C_d(i,m)$
$\overline{C}_d(i,m)$	overestimate of $C_d(i,m)$
c_h	linear unit cost of holding inventory per unit time
$C_h(i,m)$	the holding cost component of $C(i,m)$
$\underline{C}_h(i,m)$	underestimate of $C_h(i,m)$

KEY TO SYMBOLS (Continued)

$\overline{C}_h(i,m)$	overestimate of $C_h(i,m)$
$C_i(m)$	expected total cost from production point i-1 to production point i
$\underline{C}_i(m)$	underestimate of $C_i(m)$
$C_I(i,m)$	total expected cost component of $C(i,m)$ due to realization I
$C_{II}(i,m)$	total expected cost component of $C(i,m)$ due to realization II
$C_{III}(i,m)$	total expected cost component of $C(i,m)$ due to realization III
C^*	optimal total expected cost of problem
$C(i,m)$	total expected cost given i units in inventory and m units produced (deterministic case) or m units in inventory after production (stochastic case) and optimal individual unit disposal
$C\ (i,m,S)$	total expected cost given i units in inventory and m units produced (deterministic case) or m units in inventory after production (stochastic case) when production is required and disposal times are specified for the individual unit disposal policy
$\underline{C}(i,m)$	underestimate of $C(i,m)$
$\overline{C}(i,m)$	overestimate of $C(i,m)$
D	maximum order-size
$\overline{D}$	mean order-size
d_0	initial order-size
E	expected number of units produced when production is required for $m = 1$
$f(t)$	order interarrival time density function
$F(t)$	order interarrival time cumulative distribution function
$\overline{F}(S_i)$	probability that an order does not occur before S_i i.e., $1-F(S_i)$
$G(c_d)$	underestimate of the probability of disposal of a unit

KEY TO SYMBOLS (Continued)

i	particular order or unit under consideration
K_d	fixed cost per disposal
K_h	one-time fixed inventory holding cost
K_p	fixed cost per production setup
L_i	cost portion numerator divided by denominator of the disposal time formula under individual unit disposal
m	production level in orders (deterministic case) or units (stochastic case)
mcpu	minimum cost per additional unit produced
m_h	an inventory level greater than m_1 and less than or equal to m_u
m_L	lower limit on m^*
m_u	upper limit on m^*
m_1	lowest value of m such that $C(m_1-1,m_1-1) < C(m_1,m_1)$
m^*	optimal production level (deterministic case) or optimal inventory level (stochastic case)
$P_f(m)$	probability of future production given m units produced (deterministic case) or m units in inventory (stochastic case) at an order point
$\underline{P}_f(m)$	underestimate of $P_f(m)$
$\overline{P}_f(m)$	overestimate of $P_f(m)$
P_∞	probability of no future order occurring given the order process is at an order point
$p(x)$	order-size distribution
RHS	right hand side
S	vector of disposal times
S_i	disposal time i-th unit
S_1	disposal time for first unit

KEY TO SYMBOLS (Continued)

S_2	disposal time for second unit
$\overline{S}$	overestimate of disposal times
t	time
t_m	mode of the order interarrival time distribution
t_1	order interarrival time for first reorder
x	variable indicating quantity that is produced or disposed at one time
y_0	expected number of units necessary to produce in order to satisfy the initial order
y_i	expected number of units necessary to produce in order to satisfy the i-th reorder
$\Delta C_d(i,m)$	difference between $C_d(i,m+1)$ and $C_d(i,m)$
$\Delta C_h(i,m)$	difference between $C_h(i,m+1)$ and $C_h(i,m)$
$\Delta P_f(m)$	difference between $P_f(m+1)$ and $P_f(m)$

Abstract of Dissertation Presented to the Graduate Council
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PRODUCTION - INVENTORY DECISIONS UNDER ORDER-OCCURRENCE UNCERTAINTY

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This dissertation examines the problem of making optimal inventory decisions under uncertain order-occurrence. It is assumed that products can be analysed separately and that each product has one storage location. Orders can be viewed as either deterministic or stochastic in size. In the stochastic order-size case, order-sizes are assumed to be determined from a stationary probability distribution. Each time an order is received, there is a positive probability, less than unity, that no additional order occurs. When another order does occur, a stationary interarrival time distribution is used to describe the interarrival times. Order sizes and interarrival times are assumed to be independent. No backlogging is permitted, and production and disposal leadtimes are zero.

There are three types of relevant costs: production, holding, and disposal costs. Inventory disposal must be considered, because the order process may terminate while inventory is still on hand. Production costs consist of a setup cost and a linear variable cost. Holding costs include both a one-time fixed cost associated with a policy of holding inventory, and a per unit-time cost. Disposal costs are similar to production costs, and consist of a fixed cost and a linear variable cost. Production plus disposal unit costs must be nonnegative.

Four basic models are examined in this dissertation: (1) deterministic order-size-zero-fixed-disposal-cost, (2) stochastic order-size-zero-fixed-disposal-cost, (3) stochastic order-rize-positive-fixed-disposal-cost, and (4) deterministic order-size-positive-fixed-disposal-cost. Problem 4 is treated as a special case of Problem 3. All four problems are formulated as continuous time models that can be solved iteratively. Optimal disposal times are determined by differentiating the expected total cost formula and solving the necessary condition in the classic manner. Results are presented that show that only integer inventory levels need to be considered to determine the optimal inventory policy; and, an upper bound on the optimal inventory level is determined. By examining every inventory level up to the upper bound, a least-cost solution for a particular type of disposal policy can be found. In the zero-fixed-disposal-cost problem, the optimal cost can be computed, because it is shown that individual unit disposal is optimal. In the positive-fixed-disposal-cost problems, the disposal policies are limited to either individual unit disposal, or disposal of all remaining inventory, and suboptimal solutions can be obtained.

The solution algorithm, based on theoretical results, is significantly faster than a previously suggested method of solution using dynamic programming. For an example solved using the first model, the algorithm is 1000 times faster, and for a more complex example solved using the third model, the algorithm is 45 times faster. The speed depends on the number of possible inventory levels. As a byproduct, the algorithm computes the expected total costs of using lower than optimal inventory levels.

CHAPTER I

INTRODUCTION

Introduction

This dissertation examines optimal policies for the production-inventory problem with uncertain future orders. This is a pervasive problem in industry, where many products are subject to uncertain future orders. The uncertainty may be in either size, or occurrence. Uncertain order-occurrence exists if there is a positive probability, less than unity, that the current order is the last order that will occur. Inventory of the item may be disposed of sometime after the most recent (hereafter, the "previous") order, so that it will not be necessary to hold an item forever, e.g. if there is stock on hand and no future order occurs. Uncertain order occurrence is especially prevalent in jobshops.

Orders for products in fields that have (a) rapid technological progress, (b) large changes of taste that affect demand, or (c) numerous competitors in the market, are especially subject to uncertain order occurrence. Examples of products in these types of fields are:

a. specialized electronic parts,such as tubes used only in certain model television sets or military equipment,

b. government orders for spare parts, clothes, rifles, aircraft spare parts,

c. automobile replacement parts,

d. electronic machinery,

e. fashion clothing,

f. custom printing.

Jobshop firms can follow two extreme policy types. In the first, an item is produced only after the order arrives. This policy either implies that (a) the firm is so pessimistic about the probability of an order occurring in the near future that it does not believe items are worth inventorying for the potential orders, or that (b) the cost structure is such that it is not optimal to carry inventory. Under a no-inventory policy, by definition there will be no excess inventory. In contrast, when inventory is stocked there ordinarily will be excess inventory. Nonetheless, carrying inventory for possible future orders may eliminate one or more costly production setups. This will result in a net savings if the expected inventory costs are less than the expected reduction in future production setup costs. Both a no-inventory and a positive-inventory policy result in decisions that do not consider the order-occurrence uncertainty. Yet, as shown by Pierskalla (16), such uncertainty can indeed have an impact on the optimal decision in this type of problem.

If rewards for operating the jobshop in a less costly manner outweigh the sanctions associated with scrapping inventory, then the firm should investigate stocking inventory for possible future orders. Because of the uncertainty of future orders, the firm needs to consider the possibility of holding a nonoptimal level of inventory for a particular realization of the order process. When inventory and disposal costs are small, or when there is a high probability of future orders, a detailed analysis of uncertainty may not be justified. Inventory may be stocked as if there were no order-occurrence uncertainty.

As herein defined, the optimal inventory policy is that which has the least expected cost. This definition implies risk-neutrality on the part of the firm. In addition, it will be assumed that (a) the firm chooses to meet demand with no backlogging, and (b) parameters are constant over time.

Under conditions of order-occurrence uncertainty, when an order for an item is received, there is a positive probability less than unity that future orders for the item will be received.

Hence, the firm is faced with the decision of how much inventory to stock. The following questions must be answered by the firm in order to determine the optimal policy:

a. Is it optimal to stock inventory?

b. If it is optimal to stock inventory, how many units should be produced?

c. If future orders are not realized, what is the optimal disposal policy?

d. How sensitive is the expected total cost to alternative policies?

e. How sensitive is the optimal policy to various costs?

Because the parameter values may not be precisely known in a particular application, or because the model presented is a simplification of reality, sensitivity analysis may provide a useful insight into the problem and factors that affect the optimal solution. The United States Navy and the National Science Foundation have considered this area important enough to support research on these types of problems (3, 16).

The specific jobshop problem to be considered is described below. It is assumed that there is an infinite number of possible future orders. When each order occurs, there is a positive probability, less than one, that it is the last order. For each order, the order quantity is determined from a stationary demand distribution. Interarrival times between successive orders are independent random variables, where each order interarrival time is selected from a defective cumulative distribution function (DCDF). A DCDF is a cumulative distribution in which there is a positive probability that the interarrival time will be infinite, or that there will be no future orders. Further assumptions about the problem are listed below:

a. Demand must be met when it occurs.
b. Leadtime for disposal is zero.
c. The production rate is infinite.
d. Each item may be treated separately.
e. Only one inventory location is allowed.
f. Initial inventory is zero.

The total cost equation is built up from three types of costs: production, disposal, and holding costs. Each type of cost consists of a fixed component and a variable linear component for positive quantities of production, disposal, and inventory. The cost is zero otherwise. Variable production and holding costs increase as the quantity increases. If the variable disposal cost per unit is positive, then variable disposal cost increases as the number of units disposed of increases. For negative per unit disposal costs, the opposite is true. The fixed holding cost of inventory, K_h, represents the fixed costs associated with the capability of stocking inventory, such as fixed warehouse expenses, which occur regardless of the inventory level. If inventory is never stocked, then the fixed cost of inventory is avoided. Discounting of costs is not considered.

The purpose of this dissertation is to determine the optimal policy for the production-inventory problem with uncertain future orders. An algorithm is developed that solves this type of problem. The algorithm is based on conditioning on the order arrival and disposal events, and examination of the inventory level. Lemmas are developed that limit the candidates for the optimal inventory level. Differential calculus, applied in the classical manner, is the method used for determining the optimal disposal times.

Previous Studies

Previous studies on the uncertain order-occurrence problem have been limited to Jesse (8, 9, 10, 11, 12). Veinott (18) has reviewed the mathematical inventory theory literature prior to 1966, and found no research on the uncertain order-occurrence problem. For certain order-occurrence, several authors have developed theorems that limit the candidates for the optimal policy. Beckman (2) presents a certain order-occurrence problem in which the demand is described by interarrival and order-size distributions. Obsolescence of inventory is a problem related to uncertain order-occurrence, and the two most relevant papers are summarized below.

Jesse (8, 9, 10, 11, 12) formulates the uncertain order-occurrence problem as a finite horizon, discrete-time dynamic program.

The maximum number of potential orders is fixed and finite. Two probability distributions are used to describe the demand-generating process. It is assumed that the number of periods that elapse between the receipt of successive orders for an item is determined from a defective probability mass function (DPMF), and the size of each order is described by a second probability mass function. There are fixed and variable costs of production, and disposal and variable holding costs. For the case of deterministic order sizes with piecewise-linear production-disposal costs, and concave increasing holding costs, Jesse shows that the optimal inventory level will equal the sum of a number of consecutive future order quantities. In the stochastic order-size problem, an extensive sensitivity analysis is presented. The analysis of an example shows that as inventory become more expensive to hold, either by increasing the holding or disposal costs, or by decreasing the probability of future orders, less inventory is held in the optimal policy. Jesse also solves an infinite potential future-order problem by assuming a large, fixed number of potential future orders.

In the infinite reorder case numerical methods, particularly ones using dynamic programming, suffer because the theorems limiting possible policies do not limit the time horizon that needs to be considered. This requires an infinite number of stages in dynamic programming formulation. Because the interarrival time between orders is assumed to have a finite number of possible values, the selection of a finite number of potential reorders requires a finite but large number of stages. To assure convergence of cost and policy, a large number of potential reorders must be considered. This could be very expensive. For example, in Jesse (10) the unbounded horizon,stochastic demand problem required several times more computer time to achieve cost convergence with 1% of the optimum than a four potential reorder problem; but, the optimal policies are the same.

E. Zabel (20) and G. Eppen, F. Gold, and P. Pashigian (4) develop theorems relating to the finite planning horizon inventory version of the certain order-occurrence problem. They assume an N-period problem. Demand for a period is deterministic, but can vary by periods. Holding costs are deterministic per unit time costs that can vary by periods. Fixed costs of ordering or production setup are allowed to vary from period to period. Unit ordering costs can also vary by period.

Zabel's (20) first three theorems are proved in Wagner and Whitin (19) for a constant setup or ordering cost. The three theorems state that:

Theorem Z1. ordering will not occur while the inventory level is positive,

Theorem Z2. order quantity is the sum of the next j future demands,

Theorem Z3. ordering inventory at time t breaks up the problem into two subproblems: before t, and t and after (regeneration).

In the fourth theorem, Zabel (20) states that:

Theorem Z4: only periods after the last known order need to be considered as candidates for ordering.

The latter is a forward planning horizon theorem.

When unit ordering costs are allowed to vary between periods, Theorems 1, 2, and 3 still hold, although Theorem 4 may not. To replace Theorem 4, Zabel (20) uses a backward recursion cost equation and develops Theorem 5, The Backward Planning Horizon Theorem, which states:

Theorem Z5: (a) only periods after the last known order need to be considered as candidates for ordering,

(b) if the inventory level is known at a particular time, then the optimal inventory level at any time before the known time is less than the known inventory level plus the demand between the time in question and the known time.

Eppen *et al.*(4) provide theorems that limit the number of calculations that are needed to find the optimal policy. If the unit ordering cost at time t is less than the unit ordering cost at some time $j < t$, plus the holding cost from j to t, condition *, then

Theorem E1. for the t_1 period problem, $t_1 > t$, the last setup does not occur before the last setup in the t period problem.

Theorem E2. if the last setup in the t period problem is at t, then in the t_1 period problem, $t_1 > t$, there is also a setup at t.

From Theorem E2, the t_1 period problem can be divided into the t-1 period problem and the t to t_1 period problem. Further proofs show that if the last setup in the t_1 period problem is before the last setup in the t period problem, then condition * is not true. This means that it is cheaper to produce the units needed to meet the demand in period t, before period t. The holding cost is not great enough to offset this difference. Lastly, Eppen *et al.*(4) show that if the fixed setup cost is monotone nondecreasing, then the t_1 period problem can be subdivided according to (b).

Beckman (2) presents a model in which demand is described by interarrival time and order-size distributions. There is a fixed ordering cost and linear production, holding, and shortage costs. Order occurrence is certain. Using dynamic programming, Beckman shows that the s,S policy is optimal because the recursive equations are convex. To solve for s and S, the Wilson lot-size formula is chosen as the initial value of S. The equations for s and S are solved iteratively. Efficiency, or even convergence, of this algorithm is not discussed.

Brown, Lu, and Wolfson (3) formulate the obsolescence problem as an infinite horizon, discrete-time dynamic program. It is assumed that the process can be in any one of n demand-generating states during each period. Associated with each state is a probability mass function that describes the demand each period that the system is in that state. If one of the states is an absorbing state with no demand, then this formulation can represent the discrete-time, uncertain order-occurrence problem. The production, holding, and shortage cost functions are unrestricted in type. Value iteration is used to obtain the optimal solution. Rapid convergence is claimed without proof.

Pierskalla (16) approaches the obsolescence problem using an N-period dynamic programming model in which the number of periods, N, is a random variable. Costs are assumed to be convex, demand is stochastic, and backlogging is allowed. Inventory policies of the s,S type are shown to be optimal. Four examples are solved to demonstrate that the optimal reorder level S can be significantly less in the random-termination case than in the known-termination case. This problem is an uncertain order-occurrence problem in which the time between potential orders is restricted to one time unit.

Additional references to related inventory problems are Arrow, Harris, and Marschak (1), Greenberg (5), Hadley and Whitin (6, 7), Mohan and Garg (13), Morse (14), Naddor (15), Veinott (17), and Wagner and Whitin (19).

Overview of the Dissertation

In discussing the production-inventory problem four cases are considered:

1. deterministic order size and no fixed cost of disposal,
2. stochastic order size and no fixed cost of disposal,
3. deterministic order size with a fixed cost of disposal,
4. stochastic order size with a fixed cost of disposal.

Each of the first two cases has a chapter devoted to it, and the last two cases are combined into a single chapter. The structure of each chapter is basically the same, except that Chapter IV includes a section devoted to case three.

The Introduction contains a description of the problem discussed in the chapter, including assumptions. The demand-generating process is described and notation is established.

In the Model Formulation section the model is formulated, and the various sequences of order arrival and disposal are examined. It is shown that simultaneous production and disposal, noninteger disposal, production, or inventory are not optimal, which limits the number of possible realizations of the inventory process.

The expected costs associated with each sequence of events or realization are developed. By summing the expected costs for all possible realizations, an expression for the total expected cost is obtained. As the inventory level increases, the number of potential realizations of the inventory process rapidly increases. This could lead to computational difficulty in solving the problem, but the model is formulated so that this does not prohibit the calculations. By realizing that the order process is the same at every order point, that is, it is regenerative, a recursive formulation is possible. This greatly simplifies the cost equation for the general case.

The Optimal Inventory Policies section presents restrictions on optimal policies. The optimal disposal times are determined from formulas obtained from the first derivatives of the total expected cost equation. These results state that for an optimal disposal time, the probability of the order-occurrence function should equal a particular value determined as a function of the inventory level multiplied by the probability that an order occurs after the disposal time. These results are presented in the Model Formulation section of Chapter IV, the deterministic order-size, fixed-cost-of-disposal results are in The Deterministic Order-Size Problem section of that chapter.

Upper and lower limits on production are determined by comparing the marginal cost and benefit of holding inventory. This is done either by using estimates of the total expected cost function, or by comparing directly the marginal cost and benefit of each unit of inventory.

The algorithms, presented in The Algorithm section of Chapters II, III and IV are iterative ones in which all possible production levels between the upper and lower bounds are examined. Because the computation of the lower bound is time consuming, the algorithms compute the expected total cost for inventory levels at each positive integer less than or equal to the upper bound. As an extra dividend of this approach is that it shows the expected total cost of all inventory levels less than the upper bound. The algorithm proceeds as follows. The best disposal times are determined for each production level. The formulas are solved in an iterative manner, wherein the initial disposal times are substituted into the total cost equation, and then recomputed. This procedure is repeated until the differences in total cost become arbitrarily small. The number of production levels to be considered is not great, because the upper bound is revised as better estimates for the total cost are computed.

To illustrate the algorithm, a numerical example is presented in the Numerical Example section. The results for several production levels are given to show the effects of holding inventory. Costs are changed to show their effect on the optimal policy. Discussion on using the policy is also included.

CHAPTER II

THE DETERMINISTIC ORDER-SIZE PROBLEM WITH NO FIXED COST OF DISPOSAL

Introduction

This chapter considers the deterministic order-size problem with no fixed cost of disposal. In formulating the problem, there is no limit placed on the number of orders that may occur. At each order point (i.e., the time an order is received) there exists a positive probability, $0 < (1 - p_\infty) < 1$, that another order will occur. Although the exact number of orders that will occur is uncertain, the size of each order, should one occur, is assumed to be known.

Let the order interarrival time distribution be described by a defective density function, $f(t)$, such that $\lim_{t\to\infty} (\int_0^t f(u)\, du) < 1$. In this particular case $\lim_{t\to\infty} (\int_0^t f(u)\, du) = 1 - p_\infty$. This implies that upon receiving any order, there is a positive nonunitary probability p_∞ that no additional orders will occur. The initial order point is assumed to occur at time zero.

The specific cost functions assumed for production, disposal, and holding of inventory are as follows:

$$\text{Production cost} \quad \left(\begin{array}{ll} K_p + c_p \cdot x & x > 0 \\ 0 & x \leq 0 \end{array} \right)$$

$$\text{Disposal cost} \quad \left(\begin{array}{ll} K_d + c_d \cdot x & x > 0 \\ 0 & x \leq 0 \end{array} \right)$$

$$\text{Holding cost rate} \quad \left(\begin{array}{ll} c_h \cdot x & x > 0 \\ 0 & x \equiv 0 \end{array} \right)$$

where K_p, K_d, K_h are constants, and x is the amount produced, disposed, or held, respectively. For this problem, all variable costs are assumed to be linear. The fixed holding cost is a one-time cost that is incurred if inventory is ever held.

In summary, the following assumptions are made for the analysis of this problem:

1. There is an infinite number of potential reorders, because the demand process is stationary.
2. The order size is one. This can be assumed without loss of generality, by appropriate changes in the cost coefficients of the holding, production, and disposal cost functions.
3. The interarrival time between successive order points is determined from a defective, unimodal density function, f(t).
4. Demand must be met when it occurs.
5. Leadtime for production and disposal is zero.
6. Each item is analyzed separately.
7. Only one inventory location is allowed.
8. Initial inventory is zero.

The model is formulated in the Model Formulation section. Initially, three inventory policies are examined: producing to stock zero units, one unit, and two units. The possible realizations of the inventory process are determined for each of these policies. The realizations are the sequence of events that can occur and cause the inventory level to decrease to zero. These events are order arrivals, disposals, or a combination of the two. To limit the number of realizations that need to be considered, an assumption and a lemma are used. The assumption states that inventory, production, and disposal quantities are integer, and the lemma states that production and disposal does not occur simultaneously when the optimal policy is followed.

The expected cost associated with each realization is determined for the particular holding, production, and disposal costs. The total expected cost is the sum of the individual cost components. A general cost formula for any inventory level is developed. Although this formula is complex, when feasible it has been simplified by using recursive expressions. The optimal disposal times are shown to be independent of the production level. This is accomplished by showing that the same formula for determining the optimal disposal times is obtained from any production level that yields an inventory of at least the amount in question.

By using the first derivative of the total cost formula, the optimal disposal times are computed in the Optimal Inventory Policies section. Using second derivatives, the result is shown to be a minimum. Because the disposal time occurs as a lower limit of the integral, and as an argument for the order interarrival distribution, numerical methods may be necessary to solve for the disposal times. For the problems presented in the Numerical Example section, only two or three iterations are required to solve for the disposal times to an accuracy of .05 time units.

Upper and lower limits on optimal production are determined by showing that the initial production cost is convex monotone increasing, future costs are convex monotone decreasing, and disposal and holding costs are monotone increasing. The monotone increasing costs are estimated in such a way that once the inventory is greater than the optimal level, the size of the estimation error increases monotonically.

By overestimating the future costs, and underestimating the initial production, holding, and disposal costs, an upper bound is established. The reverse is done to determine the lower bound.

The algorithm initially computes the cost of producing for orders as they occur. This cost is an upper bound on the optimal cost. The total expected cost is computed for each possible production level, that is, the positive integers. This is done by determining the disposal times, and then substituting the disposal times in the total expected cost formula. The disposal times are then recomputed. This is continued until the disposal times have been determined. This is a faster procedure than previous algorithms, as shown in the Numerical Example section. The bounds on production are recomputed after the cost of each production level has been established. When the cost of producing at the upper-bound level has been determined, the least-cost solution is chosen. The fixed cost of holding inventory is added to the cost of this solution, and the result is compared to producing for orders as they occur. The lesser cost solution is optimal.

A numerical example illustrates the algorithm. Parameters are varied to determine the effects on the optimal policy and cost. For the example presented, the optimal policy is relatively insensitive to changes in the parameters, except to changes in P_∞. The algorithm is much faster than previous algorithms. An interesting result is that both the mean cost and standard deviation decrease as the inventory level increases up to the optimal level. This is intuitively appealing, because as the inventory level increases, the minimum cost increases; but, the rise in cost as the number of order increases is not as fast. Because of the smaller slope, the variation is less. By attaching the correct probabilities to the number of orders, one would expect smaller variance as the inventory level increases up to the optimal level.

Formulating the Model

In this section, lemma is developed to limit the possible candidates for the optimal policy of the deterministic order-size problem with the production, holding, and disposal costs previously presented. The total expected cost formula for the problem is developed, first by developing the formulas for zero, one, and two units of inventory, and then by developing the general formula. The technique used to develop the formula is to condition the process on the sequence of orders and disposals to yield all possible realizations of the process. In this problem the inventory level is one less than the production level. Using the general formula, the cost of any positive integer production quantity can be determined. In the algorithm, the cost computed from the formula is used to determine the optimal production quantity.

The expected cost component is determined for each possible realization of the process. These costs then are used to determine total expected cost. After the receipt of each order, a production decision is made, and the disposal policy is established. Disposal times can be determined at an order point, because no new information about the process is received between orders.

Lemma 1 is presented to establish limitations on the production-disposal alternatives that need to be considered.

Lemma 1. There exists an optimal inventory policy such that production occurs only at order points at which the order cannot be filled from stock on hand.

Although formal proof is omitted, it should be remarked that it is not optimal to produce if there is a nonnegative inventory level, because of the possibility of no future orders. There is no penalty in delaying production when the inventory level is nonnegative; but there is a potential saving of production cost.

It is assumed that the optimal inventory level is an integer ≥ -1. Jesse (10) has proved this result for the discrete time case. For Jesse's proof to hold, there must be a countable number of discrete points. Because the rational numbers are countable, his proof holds for the rational numbers. Assuming the optimal inventory level everywhere is an integer is not a very restrictive assumption because there are rational numbers that are arbitrarily close to any real number. In usual inventory applications, only rational points in time would be considered for disposal because of the difficulty in using irrational numbers.

Consider a policy of producing for orders as they occur. For such a policy there is no inventory and, hence, no disposal. Thus, the expected cost of this policy depends only on the production cost and the expected number of production setups. Define $C(i,m)$ as the total expected cost given (a) i units of inventory after receiving an order but before production, (b) a policy of producing m units whenever production is required, and (c) an optimal disposal strategy. Under a policy of producing for orders as they occur, a production cost of $K_p + c_p$ is incurred each time an order is received. The total expected cost is then

$$C(-1,1) = K_p + c_p + C(0,1) \tag{2.1}$$

given an inventory level of -1. C(0,1) represents the total expected cost given zero units of inventory at an order point, and a production policy of producing one unit whenever production is required, where

(2.2) $C(0,1) = (1-P_\infty)\, C(-1,1).$

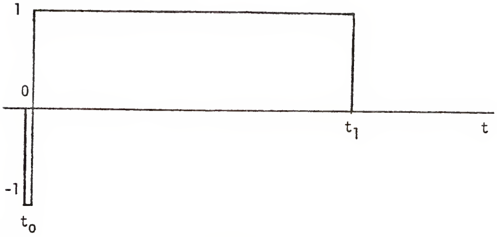
Substituting (2.2) into (2.1) and combining terms,

(2.3) $C(-1,1) = \frac{1}{P_\infty}\,(K_p + c_p).$

Assume the production policy is to produce two units whenever production is required. In this case, whenever there is one unit in inventory, two possible realizations can occur with respect to the receipt of the next order: (1) the order can be received before disposal, or (2) the unit in inventory can be disposed, because no order was received prior to the disposal time. The first realization occurs if $0 < t < S_1$, where t is the order interarrival time and S_1 is the disposal time for the unit in inventory; and the second realization occurs if $t > S_1$. These two realizations are shown in Figure 1.

For each unit of inventory there is a disposal time S_i, where i is the number of units in inventory just prior to the disposal of the ith unit. It is necessary to consider disposal, because future orders may not be realized. This feature would cause an infinite holding cost if inventory is maintained in perpetuity. The time S_i is measured from the most recent order point. That is, S_i is the time that must pass since receipt of the most recent order before the ith unit is dispensed. Because the size of an order is assumed to be unity, there is only one disposal time associated with a particular order under a given policy.

Realization I



Realization II

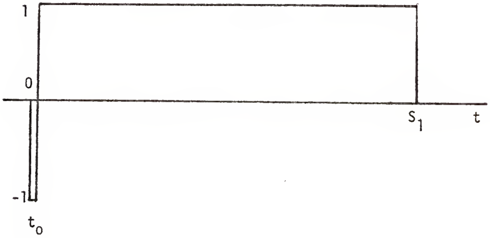


Figure 1

Possible Process Realizations for the Deterministic Order-Size Problem When $m = 1$.

The total expected cost can be written as the sum of two cost components, one associated with each realization

$$C(1,2) = C_I(1,2) + C_{II}(1,2),$$

where $C_I(1,2)$ and $C_{II}(1,2)$ are cost components associated with realizations I and II, respectively.

Define future costs to be all costs incurred after the inventory level next becomes zero. The expected cost component associated with realization I, given one unit in inventory, is then

$$(2.4) \quad C_I(1,2) = c_h \int_0^{S_1} tf(t)\, dt + C(0,2)\, F(S_1)$$

where S_1 is the length of time that must pass without an order before the unit can be disposed. There is no disposal cost in this realization, because the order occurs prior to the disposal time S_1.

For realization II the expected cost component, given one unit in inventory is

$$(2.5) \quad C_{II}(1,2) = [1-F(S_1)]\ (c_h S_1) + [1-F(S_1)]\ c_d(1) + [1-P\infty - F(S_1)]\ C(-1,2)$$

where $[1-P\infty - F(S_1)]$ is the probability that an order occurs after the disposal at time S_1.

Adding (2.4) and (2.5) yields the expected total cost of having one unit in inventory at an order point, and producing two units whenever production is required.

$$(2.6) \quad C(1,2) = c_h \int_0^{S_1} tf(t)\, dt + [c_h S_1 + c_d(1) + C(-1,2)][1-P\infty\ F(S_1)] + C(0,2)\, F(S_1) + P\infty[c_h S_1 + c_d(1)]$$

Further $C(0,2) = (1-P_\infty)\ C(-1,2)$, since at an order point, $(1-P_\infty)$ is the probability of another order arrival. Thus, an alternative expression for $C(1,2)$ is

$$(2.7)\quad C(1,2) = c_h \int_0^{S_1} tf(t)dt\ [c_hS_1 + c_d(1) + C(-1,2)][1-P_\infty - F(S_1)]$$

$$+ C(-1,2)\ (1-P_\infty)\ F(S_1) + P_\infty\ [c_hS_1 + c_d(1)]$$

Finally, whenever production is required, a production cost of $K_p + 2c_p$ is incurred. Thus, the expected total cost, given that production is required at an order point is

$$(2.8)\quad C(-1,2) = K_p + 2c_p + C(1,2)$$

We are now ready to examine the policy of producing three units whenever production is required. Whenever there are two units in inventory, three possible order arrival times can occur.

(1) $0 < t \leq S_2$, (2) $S_2 \leq t \leq S_1$, or (3) $t > S_1$, which are shown in Figure 2.

For realization I, the associated expected cost component is

$$(2.9)\quad C_I(2,3) = 2c_h \int_0^{S_2} tf(t)\ dt + C(1,3)\ F(S_2)$$

Following production, there are two units in inventory; and after the next order arrival, the inventory level is one unit.

For realization II, there is a disposal at time S_2, and the next order occurs between times S_2 and S_1. At time S_1 there is no inventory on hand, thus the associated expected cost component is

$$(2.10)\quad C_{II}(2,3) = c_h \int_{S_2}^{S_1} tf(t)\ dt + [c_hS_1 + c_d(1) + C(0,3)][F(S_1)-F(S_2)]$$

For realization III, two disposals occur, with a possible order arriving after the second disposal, at time S_1. The expected cost is

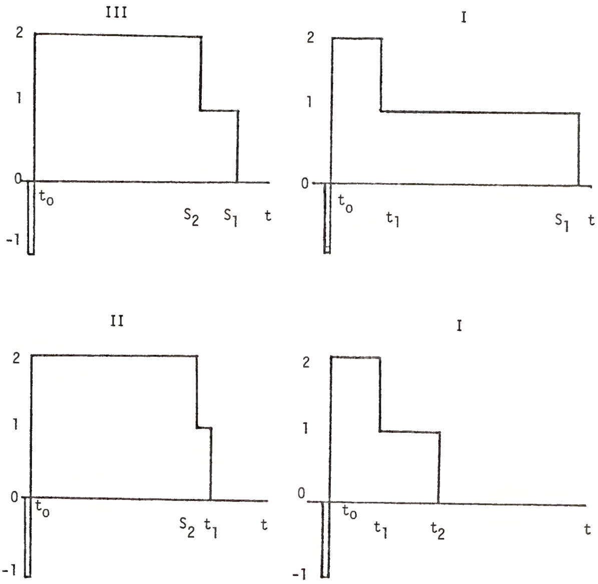


Figure 2

Possible Process Realizations for the Deterministic Order-Size Problem When $m = 2$.

$$(2.11)\quad C_{III}(2,3) = [c_h(S_2+S_1) + 2c_d(1) + C(-1,3)]\,[1 - P\infty - F(S_1)]$$

$$+ P\infty\,[c_h(S_2+S_1) + 2c_d(1) + 0]$$

The sum of (2.9), (2.10), and (2.11) yields the expected total cost of holding two units at an order point, and producing three units at each production point. This total cost is

$$(2.12)\quad C(2,3) = 2c_h \int_0^{S_2} tf(t)dt + c_h \int_0^{S_1} tf(t)dt + c_hS_1[1-P\infty-F(S_2)]$$

$$+ c_hS_2[1-P\infty-F(S_1)] + c_d(1)\,[1-P\infty-F(S_1) + 1-P\infty-F(S_2)]$$

$$+ C(1,3)\,F(S_2) + C(0,3)[F(S_1)-F(S_2)] + C\,(-1,3)[1-P\infty-F(S_1)]$$

$$+ P\infty[c_h(S_2+S_1) + 2c_d(1)]$$

For inventory levels of one and zero, the expected total costs are presented in (2.13) and (2.14):

$$(2.13)\quad C(1,3) = c_h \int_0^{S_1} tf(t)dt + [1-P\infty-F(S_1)]\,[c_hS_1 + c_d(1)]$$

$$+ C(0,3)\,F(S_1) + [1-P\infty-F(S_1)]\,C(-1,3) + P\infty[c_hS_1 + c_d(1)]$$

$$(2.14)\quad C(0,3) = (1-P\infty)\,C(-1,3)$$

Finally, a production cost of $K_p + 3c_p$ is incurred at each production point. Thus, the expected total cost given that production is required at an order point is

$$(2.15)\quad C(-1,3) = K_p + 3c_p + C(2,3)$$

For a general policy of producing m units whenever production is required, the total expected cost, as a function of the inventory level at an order point, is determined by induction to be:

$$(2.16)\quad C(-1,m) = K_p + mc_p + C(i,m)$$

$$(2.17)\quad C(i,m) = \sum_{k=0}^{i} \{k c_h \int_{S_{k+1}}^{S_k} t f(t)\, dt + [F(S_k) - F(S_{k+1})][C(k-1,m) + (i-k)c_d(1) + c_h \sum_{n=k+1}^{i} S_n]\} + P\infty\, [i\, c_d(1) + c_h \sum_{n=1}^{i} S_n] \qquad i = 0,1,\ldots, m-1$$

where $S_0 = \infty$ and $S_j = 0$.

The first term in (2.17) is the expected holding cost for all units that are in inventory when the reorder occurs. The second integral is the probability that an order occurs between the time of disposal of the k+1 unit and the potential disposal time for the k unit. Costs from the reorder point on are in the second term. The inventory level is k-1, because the inventory level prior to filling the order is k. The number of disposals is i-k. The variable disposal cost is the second term. Holding cost for those units that are disposed of prior to the first reorder is the next term. If no order occurs, the expected cost is given by the last two terms.

The problem to be solved then is

$$(2.18)\quad C^* = \min_{m} C(-1,m) - \min_{m,S} C(-1,m,S)$$

where $C(-1,m,S)$ is the expected total cost of producing m units whenever production is required, and having disposals scheduled at $S = (S_1, S_2, \ldots, S_{m-1})$.

The solution to (2.18) is the optimal policy.

Constraints on the Optimal Policy

In this section constraints on the optimal policy are determined. First, the optimal disposal times for a given production level are determined by using differential calculus in the classic manner. Next, the second derivative is examined to determine if the optimal disposal times are greater or less than the mode. Lastly, results are developed that show how an upper and lower bound on the optimal production level can be determined. These bounds are determined by using estimates of the convex and monotone increasing components of the total expected cost formula. Estimates are necessary because the optimal cost is not known, but it is used in the formula. Without the upper bound, the algorithm would not have a stopping rule. In addition, the bounds must be "good" to avoid extensive and unnecessary computations.

To investigate optimal inventory disposal, consider a production policy of producing m units at each production point. Because the fixed disposal cost is zero, individual item disposal is optimal. Up to this point it has been assumed that the optimal disposal times associated with each inventory level are followed. The optimal disposal times could depend on the inventory level. To determine if the optimal disposal times are independent of the inventory level, examine the derivative of (2.16) with respect to S_e, where $e = 1,2,\ldots,i$. This is the same as the derivative of (2.17).

Applying Leibnitz's rule to (2.17) yields

$$
\begin{aligned}
(2.19)\quad \frac{\partial C(i,m)}{\partial S_e} &= e c_h S_e f(S_e) - (e-1) c_h S_e f(S_e) \\
&\quad + f(S_e)[C(e-1,m) + (i-e)c_d(1) + c_h \sum_{n=e+1}^{i}] \\
&\quad - f(S_e)[C(e-2,m) + i\, c_d(1) + c_h \sum_{n=e}^{i} S_n] \\
&\quad + \sum_{k=0}^{i} [F(S_k) - F(S_{k+1})]\, [\frac{\partial C(e-1,m)}{\partial S_e} + 0 + 0] \\
&\quad + \sum_{k=0}^{i} [F(S_k) - F(S_{k+1})]\, [\frac{\partial C(e-2,m)}{\partial S_e} + 0 + c_h] + P_\infty c_h \\
&\qquad e = 1,2,\ldots,i
\end{aligned}
$$

Both partial derivatives on the RHS of (2.19) are zero because the inventory level is less than e. By combining terms, setting the derivative equal to zero, and solving the following result is obtained:

$$(2.20)\quad f(S_e) = \frac{c_h[1-F(S_e)]}{C(e-2)-C(e-1)+c_d(1)} \qquad e = 1,2,\dots,i$$

The RHS of (2.20) is a function of the production quantity only. The optimal disposal times are obtainable from (2.20):

$$(2.21)\quad f(S_e) = \frac{c_h[1-F(S_e)]}{C(e-2,m^*)-C(e-1,m^*)+c_d(1)} \qquad e = 1,2,\dots,i$$

Derivation of (2.21) is in Appendix A.

From (2.21) we see that the optimal disposal times are independent of the initial production quantity, provided the optimal policy is followed after time zero. Therefore, the optimal disposal time, S_e^*, is the same for any initial production quantity greater than e.

Because it was assumed that at time zero the parameters of the interarrival time distribution were known, the passage of time without an order occurring does not add any new information. Hence, we would not expect that the optimal disposal times would change depending on the realization of the process. Since order disposal times are measured from the previous order, if we have a disposal at S_k, we have waited S_k units of time and need to wait only an additional $S_e-S_k, 1 \leq e < k$, units of time without an order occurring, until the disposal at time S_e.

If (2.21) cannot be solved, then e units of inventory should not be held. This is true because the range for disposal times is bounded below by zero. If a solution to the equation of a constrained problem does not exist, then a bound is optimal. This implies that e units will never be held in the optimal policy, because zero is the bound on the disposal times. It is less expensive not to produce a unit than it is to produce the unit and immediately dispose of it.

To determine if the solution to (2.20) is a maximum or a minimum, consider the second derivative of the cost equation. The Hessian need not be examined because the off diagonal elements are zero. This implies that $\frac{\partial^2 C(i,m)}{\partial S_i^2} > 0$ for $1 \leq i \leq m - 1$ is a sufficient condition for a minimum of C(i,m).

Define $L_e = \frac{c_h}{c_d + C(e-2,m) - C(e-1,m)}$ $1 \leq e \leq m-1$, which is nonnegative because $f(S_e)$ and $[1-P -F(S_e)]$ are nonnegative.

$$(2.22) \quad \frac{\partial^2 C(i,m)}{\partial S_e^2} = \frac{\partial}{\partial S_e} [-f(S_e) + L_e \int_{S_e}^{\infty+} f(t)dt]$$

$$= - \frac{\partial f(S_e)}{\partial S_e} - L_e f(S_e).$$

For a minimum, $\frac{\partial^2 C(i,m)}{\partial S_e^2} > 0$.

Since $-L_e f(S_e) \leq 0$, then $\frac{\partial f(S_e)}{\partial S_e} \leq 0$ becomes a necessary condition for a minimum.

For a unimodal interarrival time density function, this condition will be satisfied only for values of S_e is greater than or equal to the mode, t_m.

Next C(-1,m) is investigated. If this term can be shown to be convex, or unimodal, then a stopping rule for the algorithm can be obtained, i.e. if $C(-1,m+1) > C(-1,m)$, then terminate the algorithm, because m is the optimal production level.

Analysis of the properties of $C(-1,m)$ is difficult, because of the numerous, complex terms. To simplify the analysis, it is helpful to combine terms so that $C(-1,m)$ consists of four components:

1) the immediate production cost
2) holding costs between the current production point and the next production point
3) disposal costs that occur over the same interval
4) future costs measured from the next production point on.

From (2.17), it is possible to separate $C(i, m)$ into components 2,3,4 for every positive inventory level. The initial production cost, component 1, appears in (2.16). In general, the form of $C(-1,m)$ is

(2.23) $C(-1,m) = K_p + mc_p + C_h(m-1,m) + C_d(m-1,m) + P_f(m)\ C\ (-1,m)$

where $C_h(m-1,m)$, $C_d(m-1,m)$, and $P_f(m)\ C\ (-1,m)$ are complicated expressions of the expected holding, disposal, and future costs. These three expressions can be obtained by repeatedly substituting (2.17) into (2.16) until all $C(i,m)$, $i=0,1,\ldots,m-1$ terms have been replaced by expressions using $C(-1,m)$ and holding, disposal, or production costs. The terms are grouped by type of yield $C_h(m-1,m)$, $C_d(m-1,m)$, and $P_f(m)\ C\ (-1,m)$.

Examination of $C(-1,m)$ yields the following four theorems. The proofs are in Appendix B.

Theorem 1. The immediate production cost component is convex for all positive production levels.

Theorem 2. Holding costs between the current production point and the next production point are monotone increasing in m.

Theorem 3. The expected disposal cost between the initial production point and the next production point is monotone increasing in m.

Theorem 4. The future cost component is convex monotone decreasing in the production level.

Thus, $C(-1,m)$ consists of two convex components, and two monotone increasing components.

Now define $\overline{C}(-1,m)$, and $\underline{C}(-1,m)$, as upper and lower bounds, respectively, on $C(-1,m)$. That is,

$$\underline{C}(-1,m) \leq C(-1,m) \leq \overline{C}(-1,m) \qquad \text{for all } m.$$

There are many possible formulas that can be used as under and overestimates. The formulas developed on the following pages are a compromise between accuracy of the estimates and complexity of the formulas. With more complex formulas, the error of the estimates can be made smaller, but the complexity of the estimate formulas would require more extensive computations.

Define the four components of $\underline{C}(-1,m)$ as follows:

1) $K_p + c_p m$
2) $\underline{C}_d(m-1,m)$
3) $\underline{C}_h(m-1,m)$
4) $\underline{P}_f(m)\ \underline{C}\ (-1,m)$

where

$$\underline{C}_h(m-1,m) = c_h \sum_{k=1}^{m-1} F(t_m)^{k-1} \{(m-k)[\int_0^{t_m} tf(t)dt + (1-P\infty-F(t_m))t_m]\}$$

The first component is just the immediate cost of production. The third component is

$$\underline{C}_d(-1,m) = \begin{cases} c_d(1)[P_\infty(-1)+(1-P_\infty)P_\infty(m-2)+\ldots+(1-P_\infty)^{m-2}P_\infty(1)] & c_d(1) \geq 0 \\ c_d(1)\ \{[1-F(t_m)]\ (m-1)+(1-P_\infty)\ [1-F(t_m)]\ (m-2)+\ldots \\ \quad + (1-P_\infty)^{m-2}[1-F(t_m)]\ (1)\} & c_d(1) < 0 \end{cases}$$

If the unit disposal cost is nonnegative, then $\underline{C}_d(m-1,m)$ can be underestimated by underestimating the expected number of disposals. The probability of the process terminating is an underestimate of the probability of disposal. The expression for the case $c_d(1) \geq 0$ is the disposal cost of the expected number of units disposed due to process termination. This cost is an underestimate of $\underline{C}_d(m-1,m)$. If $c_d(1) < 0$, then the expected number of disposals is overestimated in order to minimize the cost of disposal. An overestimate of the probability of disposal is $[1-F(t_m)]$, because disposals cannot optimally occur before the mode. The second expression underestimates $C_d(m-1,m)$, when $c_d(1) < 0$.

The second component underestimates the expected holding cost, because the optimal disposal times are greater than or equal to the mode. By not incurring any holding cost from t_m to S_i, an underestimate results. By definition $\underline{C}(-1,m)$ is an underestimate. Further, $\underline{P}_f(m)$ is an underestimate of $P_f(m)$, where

$$\underline{P}_f(m) = 1-P_\infty-P_\infty(1-P_\infty) - P_\infty(1-P_\infty)^2 - \ldots -P_\infty(1-P_\infty)^{m-1}$$

This is an underestimate, because $F(S_i) \leq 1-P_\infty$. Thus, the fourth component underestimates the expected future cost. The expression for the underestimate is

$$(2.24)\quad \underline{C}(-1,m) = \frac{K_p+c_pm+C_d(-1,m)+c_h(m-1)(\int_0^{t_m} tf(t)dt+t_m[P_\infty+\int_{t_m}^{\infty} f(t)dt]}{1 - \underline{P}_f(m)}$$

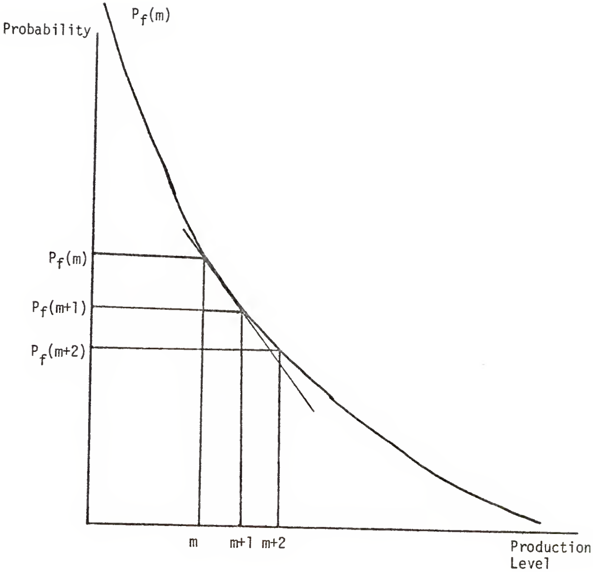


Figure 3

Probability of at Least One Production Setup In the Future Given the Production Level is m.

We form $\overline{C}(-1,m)$ in a similar manner. The four components are:

1) $K_p + c_p m$

2) $\overline{C}_h(m-1,m)$

3) $\overline{C}_d(m-1,m)$

4) $\overline{P}_f(m)\overline{C}(-1,m)$

where $\overline{S}$ is an overestimate for all disposal times, μ is the mean of the order interarrival distribution,

$$\overline{C}_h(m-1,m) = c_h \sum_{k=1}^{m-1} (m-k)\,[(1-P\infty)^k\,\mu + (1-P\infty)^{k-1}\,P\infty\overline{S})].$$

In overestimating the holding cost, note that by definition each unit is held for no more than $\overline{S}$ units of time since the previous order. If there is an order, the expected time since the previous order is less than the mean, μ, of the interarrival time distribution; if there is no order, then the expected time is less than $\overline{S}$. $(1-P\infty)^k$ and $(1-P\infty)^{k-1}P\infty$ are the probabilities of each of these two respective events occurring, and m-k is the number of units in stock. Overestimating $C_d(m-1,m)$ is accomplished by reversing the expressions derived for $\underline{C}_d(m-1,m)$. By underestimating the number of disposals when $c_d(1) < 0$, and overestimating the number when $c_d(1) \geq 0$, $C_d(m-1,m)$ can be overestimated, as shown below:

$$\overline{C}_d(-1,m) = \begin{array}{ll} c_d(1)\,\{(m-1)[1-F(t_m)] + (1-P\infty)(m-2)\,[1-F(t_m)] + \dots \\ \quad + (1-P\infty)^{m-2}(1)[1-F(t_m)]\} & c_d(1) \geq 0 \\ c_d(1)[(m-1)P\infty+(m-2)(1-P\infty)P\infty+\dots+(1)(1-P\infty)^{m-2}P\infty] & c_d(1) < 0 \end{array}$$

The overestimate of $\overline{P}_f(m)$ is obtained by underestimating $F(S_i)$ by $F(t_m)$, yielding $\overline{P}_f(m) = 1-P\infty-P\infty[F(t_m)]\dots-P\infty[F(t_m)]^{m-1}$.

The upper bound on $C(-1,m)$ is

$$(2.25)\quad \overline{C}(-1,m) = \frac{K_p + c_p m + \overline{C}_d(-1,m) + c_h(m-1)\ [(1-P\infty) + P\infty\overline{S}]}{1 - P_f(m)}$$

Utilizing $\underline{C}(-1,m)$ and $\overline{C}(-1,m)$ and Theorems 1-4, it is possible to establish upper and lower bounds on the optimal production level m^*.

To determine bounds on m^*, observe the following:

1) If we underestimate the convex monotone decreasing costs, and overestimate the increasing costs, we have a convex function whose minimum occurs at m_L, $m_L \leq m^*$. (Figure 4).
2) If we do the opposite, we have a convex function whose minimum occurs at m_u, $m_u \geq m^*$. (Figure 5).

The estimates $\overline{C}(-1,m)$ and $\underline{C}(-1,m)$ can be used to bound the slope of $C(-1,m)$. Each component of $C(-1,m)$ is bounded at each production level by the upper and lower estimates of that component. Therefore $\underline{C}(-1,m+1) - C(-1,m)$ is bounded above and below by $\overline{C}(-1,m+1)-\overline{C}(-1,m)$ and $\underline{C}(-1,m+1) - C(-1,m)$, respectively. This implies that as m increases, $m \geq m^*$, the cost estimates have larger absolute error. By having larger absolute error, this implies that if $\underline{C}(-1,m)$ increases, then $C(-1,m)$ increases faster. Because $\underline{C}(-1,m)$ is convex, then $C(-1,m)$ must increase for all values of $m > m_u$, hence the minimum of $C(-1,m)$ must occur for some value of $m \leq m_u$. We now have upper and lower bounds on both cost and production. As we learn more about the process in terms of better bounds on the disposal times, we can revise our upper and lower bounds on cost and m^*.

Without a convex total cost function, a general nonstocking criterion cannot be proven. The lower bound on the production level must be less than two, otherwise stocking will occur. Furthermore, even if the lower bound is one, stocking will occur if any production level

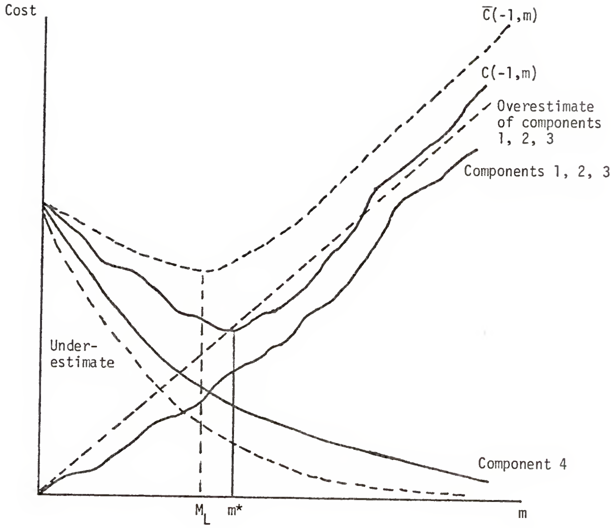


Figure 4

Cost Components and Estimates for the Deterministic Order-Size Problem

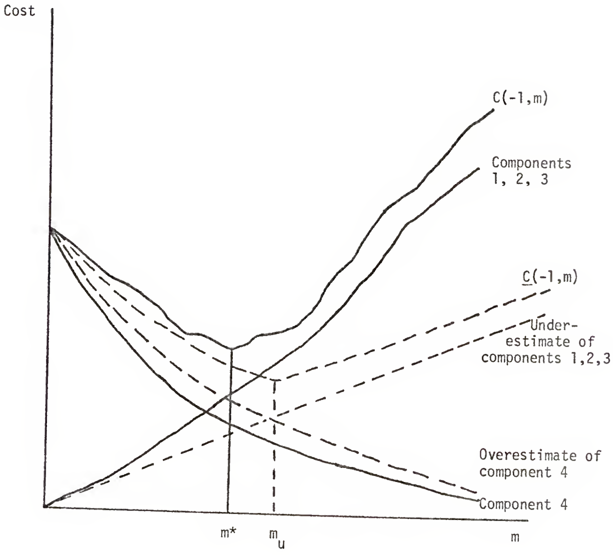


Figure 5

Cost Components and Estimates for the Deterministic Order-Size Problem Using Revised Limit, m_u

yields a cost lower than C(-1,1). The simplest production level to test is two. If $C(-1,1) > C(-1,2)$, stocking will occur. Only when the upper bound on the optimal production level is less than one can we be sure that stocking will not occur. With the results of this section it is possible to revise the expected per unit holding and disposal costs, but this is developed later.

Solution Algorithm

In this section the algorithm is presented. It has nine steps and two nested subsets of steps. Steps 4 and 5, the determination of the optimal disposal times for a particular production level, are in the inner set, or loop. The outer set, or loop, describes steps associated with determining production levels, cost, and disposal time formulas, steps 2, 3, 6, 7, 8. Step 2 calculates the cost of producing for orders as they occur, and step 9 describes how to determine the optimal solution.

The algorithm requires an upper bound on the production level. A detailed description of the algorithm is presented below.

Step 1. Assume that the upper bound on production and cost is infinite.

Step 2. Set $m = 1$ and compute $C(-1,1)$.

$C(-1,1)$ provides an upper bound on $C(-1,m^*)$.

Step 3. Set $m = m + 1$.

Step 4. Utilize (2.21) to solve for the disposal times $S_1,\dots,S_{m-1}$.

Use the costs determined from the previous production level as initial estimates of the cost for the current production level. To solve (2.21) numerical methods may be necessary. An initial value for S_i can be computed by using $P\infty$ as an approximation to $[1-F(S_i)]$, and solving (2.21) for $f(S_i)$, which determines S_i. This new value of S_i can be substituted into the RHS of (2.21), and a new value of S_i can be computed. This procedure is repeated until the change in S_i is less than some predetermined value.

Step 5. Using the disposal times calculated in Step 4, calculate $C(-1,m)$ using (2.16) and (2.17), and determine if the change of cost is less than a predetermined value. If the change is too large, go to Step 4; otherwise go to Step 6.

Step 6. Revise the least-cost solution known, if $C(-1,m)$ is less than the previous least-cost solution.

Step 7. Revise the upper bound, using (2.24) to revise the cost underestimate.

The lower bound is not used because of the time necessary to revise it. In addition, a sensitivity analysis on the production level is obtained by examining each production level.

Step 8. If $m = m_u$, then go to Step 9; otherwise go to Step 2.

Step 9. The least-cost solution for the production levels investigated, along with the corresponding disposal times, is optimal. The cost must be increased by K_h if $m > 1$.

Two critical questions arise: 1) What if there is no solution to the equations for the disposal times?, and 2) Do Steps 4, 5 converge and, if so, is the convergence rapid? If there is no solution for a disposal time S_i, then i units of inventory should not be held because this is a constrained-at-zero problem.

If a unit is held for zero units of time, we are better off not producing it, because $c_p + c_d \geq 0$. Furthermore, unit i is not held for an infinite amount of time, because this would cost more than the

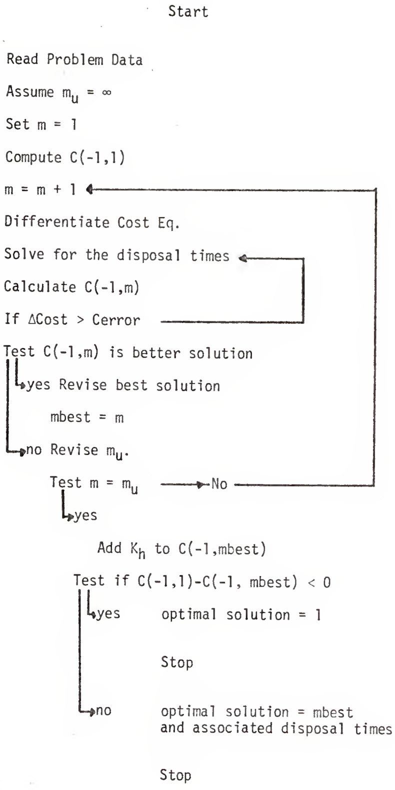


Figure 6

Algorithm Flow Chart

policy of producing for orders as they occur.

Quick convergence on S_i occurs, because the interarrival time distribution is unimodal and continuous, and because in real life S_i is computed only to a certain precision. From computational experience with the problem in the Numerical Example section only two or three iterations appear to be necessary to solve for the disposal times within .05 time units. The quick convergence occurs, because as the disposal times converge, the optimal cost converges, and vice versa. This occurs because the change in cost is caused by changes in the estimates of the disposal times. If the disposal time estimates do not change very much, then the cost estimates do not change very much. This in turn implies that the disposal times estimates do not change very much, as shown in (2.21). Newton's method can be used, if desired, because bounds on the derivative of S_i are known.

Numerical Example

To illustrate the speed of the algorithm, Example 3.5 originally presented by Jesse (10) is examined. This example was selected because dynamic programming has been the previously optimal solution technique. Jesse has refined the dynamic programming algorithm to make it considerably faster than it was previously for the deterministic order-size problem. Prior to the work presented in this dissertation, his algorithm was the fastest for this problem.

The relevant problem data is in Table 1. An infinite number of potential reorders, rather than four, is assumed. No discounting is allowed. Demand is deterministic and equals 50 units. By appropriately transforming the costs, we make the order size 1. Holding cost on a

TABLE 1

PROBLEM AND TRANSFORMED DATA

Parameter	Problem Data	Transformed Data
K_p	\$600.00	\$600.00
K_d	0	0
K_h	0	0
c_p	4.00/unit	200.00
c_d	-2.00	-100.00
c_h	.05/unit-month	2.5/unit-month
D	50 units	1
$\overline{D}$	50 units	1
p_∞	.3	.3

yearly basis is 15% of the variable production cost. The conditional interarrival times for orders are shown in Table 2. Linear interpolation is used between points. The probability of no future order, given the process is at an order point, is .3.

The costs of the policies are given below:

Production level	Cost	Upper bound m_u
1	2,667	-
2	2,015	4
3	1,938	4

A production level of four yields a higher cost.

The optimal policy is to produce three order quantities each time production is required. If the time since the last order is 33.6 time units, then dispose of one order quantity. If an additional 1.6 = 35.2 - 33.6 time units pass without an order, dispose of the remaining inventory. The total expected cost of this policy is $1,938. Using dynamic programming, the same solution is obtained: the optimal production level is three, disposal times are 33 and 35, and the expected total cost is $1,938. To determine the optimal policy required .02 seconds of computer time, compared to 17.36 seconds for the dynamic programming algorithm.

If we decide to take less risk, and produce less than the optimal quantity, the expected cost can still be less than the expected cost of producing for orders as they occur. In this example, if we produce two order quantities, two units, when production is required, then the total expected cost is $2,015, which is close to the optimal cost.

TABLE 2

CONDITIONAL INTERARRIVAL TIME PROBABILITIES

FOR NUMERICAL EXAMPLE

t	$\hat{f}(t)$	t	$\hat{f}(t)$
1	0.0000	22	0.0410
2	0.0007	23	0.0366
3	0.0023	24	0.0322
4	0.0055	25	0.0279
5	0.0100	26	0.0238
6	0.0156	27	0.0199
7	0.0219	28	0.0164
8	0.0286	29	0.0133
9	0.0351	30	0.0106
10	0.0414	31	0.0083
11	0.0470	32	0.0062
12	0.0515	33	0.0047
13	0.0550	34	0.0034
14	0.0573	35	0.0024
15	0.0585	36	0.0016
16	0.0586	37	0.0011
17	0.0574	38	0.0007
18	0.0555	39	0.0004
19	0.0526	40	0.0003
20	0.0492	41	0.0001
21	0.0453	42	0.0001

The expected discretionary cost, which is defined as all costs not related to filling the order that occurs at time zero, is examined next. These costs are the total costs less the $800 required for the initial setup and production for one unit. These costs are

m	Discretionary cost
1	$ 1,867
2	$ 1,215
3	$ 1,138

By following the optimal policy, the expected savings are $729 over the cost of the policy of producing for orders as they occur. This amounts to approximately 40% of the discretionary costs.

A computer program solved 31 problems that were cost variations of this problem in .64 seconds. Very few iterations were required to determine the disposal times. Because only a few production levels needed to be considered, the number of calculations is small. As $P\infty$ increases, or K_p becomes relatively larger than the other costs, more production levels will be examined. If $P\infty$ is close to unity and the disposal cost is small, the firm can find a good policy by using the EOQ formula for orders at the end of a period, e.g., if $P\infty > .95$, $c_d = - c_p$ and K_h, $K_d = 0$. The production level does not increase very fast as K_p increases. In the above problem when $K_p = \$2,000$, the optimal production level is five. The small change is caused by a $P\infty$ being large. In fact, there is less than 35% chance of more than three orders occurring. For $K_p = \$4,000$, the optimal production level is six.

Summary and Conclusions

In this chapter an iterative procedure has been developed for solving the single item, deterministic order quantity, production-inventory problem in which there is uncertain order occurrence. Assuming a fixed production cost, plus linear unit production, holding and disposal costs, we showed that the optimal disposal policy is individual unit disposal, and the inventory carried would always be a positive integer or zero. Simple conditions are presented for determining disposal times. Upper and lower bounds on the optimal production level are also developed. In the algorithm, the optimal disposal times and expected costs are computed for each possible production level up to the upper bound. Because of the time required to revise the lower bound, only the upper bound is used in the algorithm. A numerical example has been provided to illustrate the application of the algorithm.

The dynamic programming approach developed by Jesse (10, 12) does not appear practical in terms of actual implementation. The impracticality is due to computational reasons. Using Jesse's algorithm, solving 1000 problems similar to the example in the Numerical Example section would take 1000 times as long, or 17,360 seconds. For the algorithm presented in the dissertation, the time required would be 20 seconds. Using a computer slower than the IBM 370-165, such as a computer to which the typical jobshop might have access, would take considerably longer. A sensitivity analysis would not be practical using dynamic programming, because each set of parameters would take an additional five hours of computer time. In contrast, using the present algorithm, an extensive sensitivity analysis would take only a few minutes, much less time than is required to solve the 1000 problems by dynamic programming. Managerial decision making would be abetted by sensitivity information. Also, revisions to the optimal policy would be very expensive if dynamic programming were used, but would be inexpensive if the algorithm in the dissertation were used.

CHAPTER III

THE STOCHASTIC ORDER-SIZE CASE WITH ZERO FIXED DISPOSAL COST

Introduction

In this chapter a sequence of theorems are presented that permit the construction of an algorithm for determining the optimal, that is, the least expected cost, solution to the stochastic order-size, uncertain order-occurrence problem. This problem is a generalization of the problem examined in Chapter II. That is, when the firm receives an order for an item, there exists a probability that future orders for the same item will be received; but the order sizes of future orders are not known. Instead, the size of each order is described by a known probability distribution. The proofs to the theorems are presented in Appendix E. An expression for the total expected cost is derived by conditioning on all possible causes of inventory change, as was done in Chapter II. Using differential calculus, the optimal disposal times for a particular production level are determined in the classic manner. Estimates of the cost components of the total expected cost formula are determined from the lemmas and theorems presented. These estimates are used to construct an upper bound on the optimal inventory level. The upper bound is used to limit the candidates for the optimal inventory level, and thereby reduce the computations necessary to determine the optimal solution. This bound is a compromise between the complexity and restrictiveness of possible bound formulas. It is an efficient bound in terms of computer time because computing more restrictive bounds is time consuming, and the more restrictive bounds do not eliminate many additional candidates for the optimal inventory policy. Further discussion of the upper bound is presented in The Algorithm section. Then, an algorithm is presented that permits the actual computation of the optimal solution. The speed of the algorithm is illustrated by comparison to an example originally presented by Jesse. Jesse's example is pertinent in this context, because Jesse has revised the dynamic programming algorithm for this problem to make the algorithm considerably faster, and it was the fastest algorithm for this problem prior to the algorithm in this dissertation.

As in Chapter II, define the interarrival time as the time between successive orders, where the interarrival time is determined from a defective probability distribution. Further, assume the demand at each order point is determined from a probability mass function, $p(x)$, $x = 1,2...,D$, with mean, $\overline{D}$. Again, disposal must be considered because of the possibility of no future orders. Associated with each unit of inventory is a disposal time S_i, where i denotes the number of units in inventory just prior to the disposal. That is, immediately following the disposal there will be (i-j) units on hand, where j is the number of units disposed. The disposal times are measured from the previous order.

Listed below are assumptions used in the analysis of this problem. All assumptions from Chapter II apply, except the order-size assumption, and are denoted by *.

1. Order size is determined from a probability mass function defined on the integers from 1 to D.

2. The order interarrival distribution is independent of previous order size.
*3. There is an infinite number of potential reorders because the demand process is stationary.
*4. The interarrival time between successive orders is determined from a defective, unimodal density function, f(t).
*5. Demand must be met when it occurs.
*6. Leadtime for disposal is zero.
*7. The production rate is infinite.
*8. Each item is analyzed separately.
*9. Only one inventory location is allowed.
*10. Initial inventory is zero.

In the Model Formulation section a model is formulated by examining all possible realizations of the order-arrival process. Production and inventory levels are shown to be integers. Optimal disposal times are developed in the Optimal Inventory Policies section, by differentiating the total cost formula, setting the result equal to zero, and solving $f(S_i)$. An efficient upper bound on the optimal inventory level is developed, using a monotone increasing function that underestimates the total cost. A solution algorithm is presented in The Algorithm section, and the Numerical Example section contains a numerical example.

Model Formulation

In this section an assumption is presented that limits the possible optimal inventory levels,and therefore the production levels, for the stochastic order-size,uncertain future order-occurrence problem with no-fixed-cost-of-disposal permitted. As in Chapter II, the total expected cost formula is first developed for one unit of inventory, and then for the general case. An associated cost component is determined for each possible realization of the inventory process. The total expected cost is the sum of the cost components for each realization. Upon receipt of the initial order, enough units are produced to both fill the order, and reach the desired inventory level. Additional production takes place whenever an order cannot be met from inventory. Disposal times can be determined at each order point, because it is assumed that no new information about the process is received between orders.

The specific cost functions considered are those of Chapter II. All variable costs are assumed to be linear functions of output. The fixed cost of holding inventory is a one-time cost that is incurred for any positive inventory level. When comparing the cost of producing for orders as they occur, with a policy of carrying inventory, the fixed cost could change the optimal policy from holding inventory to not holding inventory. Results from Chapter II concerning the form of the optimal disposal policy, and the independence of the optimal disposal times and inventory levels, also apply to the stochastic order-size problem.

The Optimal Inventory Assumption establishes limitations on the production/inventory alternatives that need to be considered. Production takes place only when required by the no backlogging restriction, because the process may terminate before the additional units produced would be needed to fill orders. It is not optimal to produce and immediately dispose of units, because $c_p + c_d \geq 0$.

The Optimal Inventory Assumption is that the optimal inventory level after filling an order is an integer $\geq -D$ and is discussed in Appendix C.

Define $C(i,m)$ as the total expected cost from an order point at which i units remain in inventory after demand has been satisfied, an optimal disposal policy is followed, and the production policy, at each production point, is to produce enough units to reach an inventory level of m units.

Consider the policy of producing for orders as they occur. Thus, there is no inventory and no disposal; and, at each production point, the production level equals the order size. The total expected cost of this policy depends on the size of the current order, the expected number of orders, and the expected order size. Letting i denote the size of the current order, the immediate production cost is $K_p + ic_p$, and the expected cost for each subsequent order is $K_p + \overline{D}\, c_p$. The probability of a reorder is $1 - P_\infty$. Thus, the total expected cost of this policy is

$$(3.1) \quad C(-i,0) = K_p + ic_p + C(0,0) \qquad i = 1,2,\ldots,D.$$

$C(0,0)$ represents the total expected cost of the process, given there are zero units of inventory at the order point, and a production policy of producing for orders as they occur

$$(3.2) \quad C(0,0) = (1-P_\infty)\,[\sum_{x=1}^{D} p(x)\, C(-x,0)] = (1-P_\infty)\, C(-\overline{D},0).$$

By substituting (3.2) into (3.1), adding and subtracting $\overline{D}\, c_p$,

$$(3.3) \quad C(-i,m) = \frac{1}{P_\infty}\,(K_p + \overline{D}\, c_p) + (i - \overline{D})c_p \qquad i = 1,2,\ldots,D$$

This is the expected cost of producing for orders as they occur.

Consider the policy of producing to satisfy the order plus one additional unit for inventory. Three possible realizations of the order process can occur

I) An order for more than one unit is received before disposal

II) An order for one unit is received before disposal

III) No order occurs before the disposal at time S_i.

These realizations are shown in Figure 7.

The component of total expected costs that is associated with Realization I is

$$(3.4)\quad C_I(1,1) = \sum_{x=2}^{D} p(x) \{c_h \int_0^{S_1} tf(t)dt + [K_p + xc_p + C(1,1)]F(S_1)\}$$

There is no disposal in this realization. Production occurs again because of the order size, and future costs include all costs from the next order point. The same production policy is followed at all production points.

For Realization II the expression is similar to that for Realization I, except that production does not occur at the next order point, because the inventory level is zero, and not negative as in realization one. Below is the cost of Realization II

$$(3.5)\quad C_{II}(1,1) = p(1)[c_h \int_0^{S_1} tf(t)dt + C(0,1)\ F(S_1)].$$

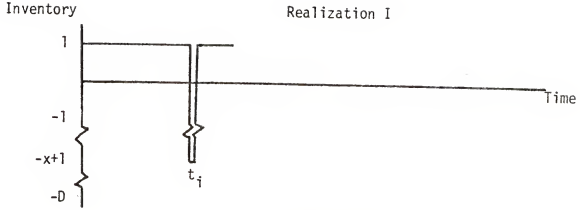
In Realization III the expected holding, disposal, and future costs are

$$(3.6)\quad C_{III}(1,1) = \overline{F}(S_1)(c_h S_1 + c_d) + [\overline{F}(S_1) - P\infty] \sum_{x=1}^{D} p(x)\ C(-x,1)$$

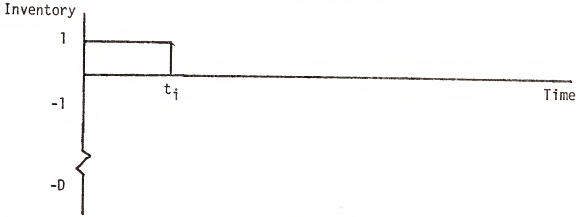
where $\overline{F}(S_1) = 1 - F(S_1)$, and $\overline{F}(S_1) - P\infty$ is the probability that an order occurs after the disposal at S_1.

Adding (3.4), (3.5), and (3.6) gives the total expected cost of the production policy of producing for the order plus producing one unit for inventory at each production point

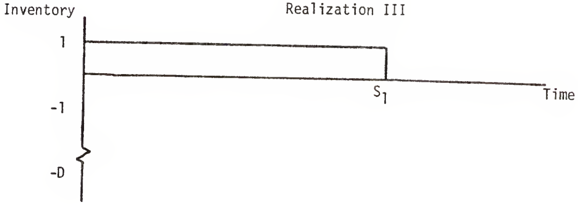
$$(3.7)\quad C(1,1) = \sum_{x=2}^{D} p(x)\{c_h \int_0^{S_1} tf(t)dt + [K_p + xc_p + C(1,1)]\ F(S_1)\}$$



Realization I



Realization II



Realization III

Figure 7

Possible Order Realizations in the Stochastic Order-Size Problem When The Inventory Level Equals One

$$+ p(1)[c_h\int_0^{S_1} tf(t)dt + C(0,1)\ F(S_1)]$$

$$+ \bar{F}(S_1)[c_hS_1+c_d(1)] + [-P\infty + \bar{F}(S_1)] \sum_{x=1}^{D} p(x)\ C(-x,1)$$

where $C(0,1) = (1-P\infty)\ C(-D,1) = (1-P\infty)[K_p+(\bar{D}+1)c_p + C(1,1)]$. The analytical solution for C(1,1) is presented in Appendix D.

The formula C(i,m) becomes more complex as the inventory level increases, because the number of terms increases quite fast, depending on the number of realizations for the particular inventory level. That is, as m increases by one unit, the number of possible order and disposal combinations that reduce the inventory level to a nonpositive integer at least doubles. This is because an order for one unit or a disposal can occur when there are m + 1 units in inventory, and would result in m units in inventory. All realizations that can occur with m units initially in inventory are still possible. This results in twice as many realizations for the m + 1 unit inventory level as for the m unit inventory level. Because the maximum order-size is greater than one, other inventory levels, less than m, can occur if an order is received when the inventory level is m + 1. For each of the other inventory levels, there are additional realizations. These additional realizations result in more than doubling the number of realizations for the (m+1)-unit inventory level as for the m-unit inventory level. The initial production cost is $K_p + ic_p + mc_p$, where m is the desired inventory level. For this problem the total expected cost is

$$(3.8)\quad C(-i,1) = K_p + (i + 1)c_p + C(1,1) \qquad i = 1,2,\ldots,D$$

because the desired inventory level is one.

For any inventory level $1 \leq i \leq D$, there are three types of realizations:

1. There is an order whose size exceeds the inventory level and occurs before S_i.
2. There is an order whose size is less than or equal to the inventory level,and occurs before S_i.
3. There is no order before the first disposal.

Consider Figure 8. Looking at the time the first order occurs, the expected cost until the next order for Realization I is

$$(3.9) \quad ic_h \int_0^{S_i} tf(t)dt + \sum_{x=i+1}^{D} C(i-x,m)p(x)\ F(S_i)$$

The future cost is determined by realizing that production is required for the first reorder, and the resulting inventory level is negative.

For Realization II the expected cost is

$$(3.10) \quad ic_h \int_0^{S_i} tf(t)dt + F(S\ )[\sum_{x=1}^{i} p(x)\ C(i-x,m)].$$

In Realization III the terms are more complex,because the first order can occur after several disposals. Assume the first order occurs between the disposal times for the (k+1)th and kth units, S_{k+1} and S_k. The expected cost is

$$(3.11) \quad c_h k \int_{S_{k+1}}^{S_k} tf(t)dt + \sum_{n=k+1}^{i} (c_h S_n + c_d)[F(S_k) - F(S_{k+1})]$$

Given that an order does not occur before S_i, the expected cost is

$$(3.12) \quad \sum_{k=0}^{i=1} \{kc_h \int_{S_{k+1}}^{S_k} tf(t)dt + [1-F(S_{k+1})]S_{k+1}c_h\} + ic_h \int_0^{S_i} tf(t)dt$$

$$+ \sum_{k=1}^{k=0} [1-F(S_k)]c_d(1) + \sum_{k=1}^{i+1} \sum_{x=1}^{\min(D,i-1)} \{p(x)C(k-1-x,m)\ \ [F(S_{k-1})-F(S_k)]\}$$

The argument k-1-x is obtained by recognizing that there are k-1 units in inventory if there are disposals at S_k, $S_{k+1},\ldots,S_i$, and an order for x units occurs after S_k, but before S_{k-1}. The term on the right

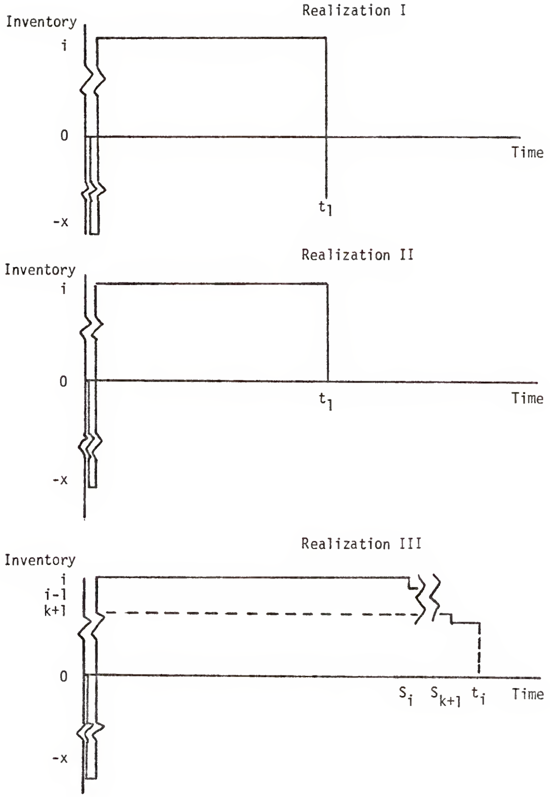


Figure 8

Possible Order Realizations in the Stochastic Order-Size Problem When the Inventory Level Equals i indicates the probability of such an occurrence.

The sum of the future cost components of (3.9), (3.11), and (3.12) can be expressed as

$$(3.13)\quad \sum_{k=1}^{i+1} \sum_{x=1}^{D} [p(x)\, C(k-1-x,m)] \quad [F(S_{k-1}) - F(S_k)]$$

The total expected cost is then the sum of (3.9), (3.11), and (3.12).

$$(3.14)\quad C(i,m) = \sum_{k=0}^{i=1} \{kc_h \int_{S_{k+1}}^{S_k} tf(t)dt + [1-F(S_{k+1})]S_{k+1}c_h\} + c_h i \int_0^{S_i} tf(t)dt$$

$$+ \sum_{k=1}^{i} [1-F(S_k)]c_d(1)$$

$$+ \sum_{k=1}^{i+1} \sum_{x=1}^{D} p(x)\, C(k-1-x,m) \quad [F(S_{k-1}) - F(S_k)]$$

where $S_0 = \infty -$, $S_{i+1} = 0$ $\qquad i = 0,1,2,\ldots,m$

$$(3.15)\quad C(i,m) = K_p + c_p(m-i) + C(m,m) \qquad i = -1,-2,\ldots,-D.$$

For this problem the total expected cost is

$$(3.16)\quad C(-i,m) = K_p + ic_p + mc_p + C(m,m).$$

The problem of interest is

$$(3.17)\quad C^* = \min_m C(-i,m) = \min_{m,S} C(-i,m,S),\ S=(S_1,S_2,\ldots,S_m),\ i=1,2,\ldots,D$$

where C(-i,m,S) is the cost C(-i,m), except that the disposal times are specified, and are not necessarily optimal.

By estimating the optimal values of the disposal times and solving $C(-i,m,\overline{S})$, the estimates of the disposal times can be reestimated. Using the algorithm presented in The Algorithm section,the disposal times are reestimated until C(-i,m) has been determined. This part of the algorithm is repeated until all candidates for the optimal inventory level has been examined. The $\min_m C(-i,m)$ can be determined, and the problem is solved.

Optimal Inventory Policies

Constraints on the optimal policy are presented in this section. These constraints consist of two types: (1) constraints on the optimal disposal times, and (2) constraints on the optimal inventory level. Using differential calculus in the classical manner, the optimal disposal times are determined from the total expected cost formula. The result is analogous to the deterministic result. The optimal inventory level is bounded above by using the theorems and observations about the total expected cost formula. Constructing the upper bound, that is used to reduce the computations necessary to determine the optimal solution, is done by a method that is similar to the method used for the deterministic case, except that future costs cannot be shown to be convex. This difficulty forces a modification to the method of constructing the upper bound. This is an important distinction,because the upper bound does not appear to be as close to the optimal inventory level for several examples as in the deterministic case. By introducing more candidates for the role of optimal inventory level, the amount of computation to determine the actual optimal level is greatly increased.

By similar reasoning to that in Chapter II, it is shown that disposal times do not depend on the production quantity, where disposal times are defined as zero for demands greater than the number of units in inventory. Further, all the information is known at the order point, and one would not expect the disposal times to vary in accordance with the realization. Since disposal times are measured from the previous order, there is no confusion if several disposals occur in the absence of an order. It has been assumed that individual unit disposal is optimal in this formulation. By allowing individual unit disposal, any disposal policy can be represented by an appropriate choice of disposal times. Individual unit disposal is optimal, just as in the deterministic case, because there is no fixed disposal cost, so there is nothing to be saved by combining disposal of individual units. Without any offsetting savings, there is no reason to deviate from the optimal individual disposal times.

For an inventory policy of holding m units, a necessary condition for the optimal disposal times is determined in the Appendix E. Individual disposal is optimal, because there is no fixed disposal cost. Differentiating the total expected cost function (3.16) with respect to S_i, setting the derivative equal to zero, and solving yields

$$(3.18)\quad f(S_i) = \frac{C_h[1 - F(S_i)]}{c_d(1) + \sum_{x=1}^{D} p(x)[C(i-1-x,m) - C(i-x,m)]} \qquad i \le m$$

The result is a generalization of the deterministic case. If D equals one, then (3.18) is the optimal condition for the deterministic case.

A computationally convenient, and easily derived upper bound on the optimal inventory level is established below. The bound reduces the number of inventory levels that have to be examined to determine the optimal inventory level.

To find the upper bound, a monotone increasing function that is less than C(i,m) is compared to the known least-cost solution, cbest. When the monotone increasing function equals the cbest, the bound is established. All inventory levels greater than the upper bound, m_u, cost more than cbest.

The formulas for holding and disposal costs, $C_h(m,m)$ and

$C_d(m,m)$, are given below:

$$(3.19)\quad C_h(m,m) = \sum_{k=0}^{m-1} kc_h \int_{S_{k+1}}^{S_k} tf(t)dt + [1-F(S_{k+1})]\, S_{k+1}c_h$$

$$+\ mc_h \int_0^{S_m} tf(t)dt$$

$$+ \sum_{k=1}^{m+1} \sum_{x=1}^{D} [p(x)\, C_h(k-1-x,m)]\, [F(S_{k-1}) - F(S_k)]$$

where $C_h(i,m)$ is the cost component for holding inventory until the next production point of $C(i,m)$;

$$(3.20)\quad C_d(m,m) = \sum_{k=1}^{m} [1-F(S_k)]c_d(1)$$

$$+ \sum_{k=1}^{m+1} \sum_{x=1}^{D} [p(x)C_d(k-1-x,m)]\, [F(S_{k-1}) - F(S_k)]$$

where $C_d(i,m)$ is the disposal cost component until the next production point of $C(i,m)$.

To determine if a disposal time is a minimum or a maximum, the Hessian should be investigated. It is apparent from (3.18), however, that all nondiagonal elements of the Hessian are zero. Thus, only the main diagonal elements need to be shown to be positive to guarantee a minimum. The proof is the same as in Chapter II, with a slight modification to allow for stochastic demand. Thus, the optimal disposal times must be greater than or equal to the mode of the interarrival distribution.

The formulas for the disposal times are independent of the past inventory, or production levels. They depend only on the optimal cost of inventory levels that can result from the reduction in the inventory level caused by a single order occurrence. If (3.18) cannot be solved, then the inventory level in question is greater than the optimal production level. This is because the disposal times are constrained from below by zero. In a constrained optimization problem, if the solution is not in the allowed range, the solution must be at an end point. In this case the end point is zero. A disposal time of zero indicates that the unit should not be produced, because it is less expensive not to produce the unit than it is to produce the unit, and then immediately dispose it.

Before the upper bound is discussed, the monotonicity of the cost components is established. In the development of the upper bound, the monotonicity is an essential property that is used to guarantee finite termination of the algorithm.

Theorem 1. The current production cost is convex monotone increasing for all positive production levels.

Theorem 2. Holding costs between one production point and the next production point are monotone increasing in m.

Theorem 3. The expected disposal cost between two production points is monotone in m. If c_d is positive, the cost is monotone increasing. If c_d is negative, the cost is monotone decreasing, but the sum of the unit production cost and the unit disposal cost is monotone increasing.

From Theorems 1, 2, and 3, it follows that the total expected cost from one production point to the next is monotone increasing in m. The initial production, holding, and disposal costs from one production point to the next, $C_i(m)$, where i is the number of the ending production point,can be computed for each inventory level. By comparing the sum to the least-cost solution, cbest, a stopping rule for the algortihm can be determined. If the sum is greater, $C_i(m)$ > cbest, there is no higher inventory level that can yield a lower cost. If the opposite is true, then the next higher inventory level needs to be investigated. The monotonicity is used to establish that once the sum $C_i(m)$ is greater then cbest, $C_i(m+j)$, $j = 1,2,...$ is algo greater than cbest. This is shown in Figure 9.

While this procedure is relatively simple, the bound obtained is not very good,because many inventory levels higher than the optimal values are included. This is not surprising,because all costs occurring after the next production point are ignored. To obtain a better upper bound these costs should be considered. The difficulty in obtaining a better upper bound is that the probability of future production is a monotone decreasing function, and may destroy the monotonicity of the sum of the cost components.

Theorem 4. The probability of future production is monotone decreasing in m, given m units in inventory at an order point, $P_f(m)$.

This difficulty in obtaining a better upper bound can be avoided if a large m is chosen that is guaranteed to be greater than m*, the optimal inventory level. Because $P_f(m)$ is monotone decreasing, selecting a large m as a bound, m_u, underestimates the probability of future production. Because the cost between production points is an underestimate, the product of the estimated cost and probability of production is also underestimated. Although this procedure for obtaining a better upper

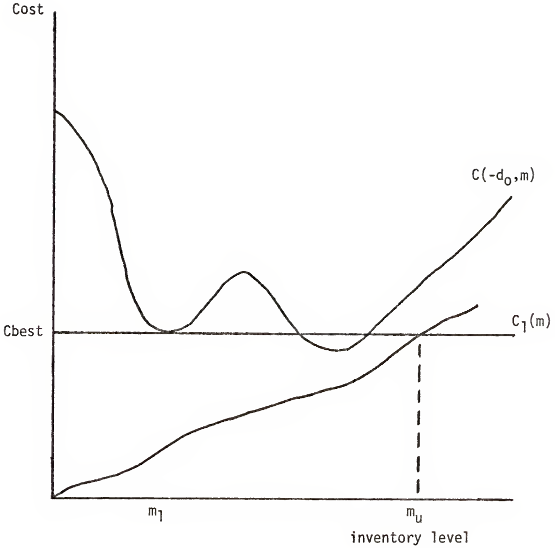


Figure 9

Determination of Upper Bound Using $C_1(m)$

bound may appear difficult, it can be done rather easily as shown below.

Initially select an upper bound. This upper bound could be ∞, or derived from warehouse, capital, or production constraints. If $C(i,m) < C(i,m-1)$, $i = -1,-2,\dots,-D$; that is, when the total cost function is decreasing, the calculation of the upper bound is not necessary, because the next higher inventory level must be investigated. The bound becomes important only when the expected total cost increases.

For some inventory level m_1 the expected total cost will be greater than the next lower inventory level, otherwise an infinite amount of inventory would be the least-cost solution, and would have an infinite total expected cost. The total expected cost from the initial production point to the next production point is

(3.21) $C_1(m_1) = K_p + (y_0 + m_1)c_p + C_h(m_1,m_1) + C_d(m_1, m_1)$

where y_0 is the initial order size, and y_i in general is the expected number of units that must be produced to satisfy the order causing production.

$C_h(m_1,m_1)$ is the holding cost component of $C(m_1,m_1)$ between the first and second production points.

$C_d(m_1,m_1)$ is the disposal cost component of $C(m_1,m_1)$ between the first and second production points.

These two components can be determined during the algorithm as they are used to compute the total expected cost.

The total expected cost of the problem is

(3.22) $C(-d_0,m) = C_1(-d_0,m)+P_f(m)C_2(y_1,m) + [P_f(m)]^2\, C_3(y_2,m) + \dots$.

The expected number of production points is $1/[1-P_f(m)]$. To construct a monotone increasing function that underestimates (3.22) for inventory level greater than m_1, each part of (3.22) can be underestimated. The total expected cost for any higher production levels has exactly the same form as (3.22), with m_h replacing m_1, $m_h > m_1$. Here, $C_1(m_1)$ underestimates $C_1(m_h)$. To underestimate the other $C_i(m_h)$, we use the fact that y_i is less than or equal to -1. The underestimate of $C_i(m_h)$ is defined as

$$(3.23)\quad \underline{C}_i(m_h) = K_p + (m_1+1)c_p + C_h(m_1,m_1) + C_d(m_1,m_1), \qquad i = 2,3,\ldots .$$

This is an underestimate because all three components as monotone increasing functions are underestimates of the initial production, holding, and disposal components of $C_i(m_h)$. If $p_f(m_h)$ can be underestimated in such a way that as m_h increases, a monotone increasing underestimate of the total expected cost is obtained. Then, the bounding procedure can be used. The formula for the underestimate is

$$(3.24)\quad \underline{C}(-d_0,m_h,m_u) = C_1(m_1) + \underline{P}_f(m_u)\,\underline{C}_2(m_1) + [\underline{P}_f(m_u)]^2\underline{C}_3(m_1) + \ldots .$$

The nonstocking criterion cannot be proved, because the total cost formula has not been proven convex, but, the same can be said about the stochastic case as was said about the deterministic case. If m_u is zero, then stocking will not occur. If $C(-i,1) < C(-i,0)$, $i = 1,2,\ldots,D$, then stocking will occur.

The Algorithm

The algorithm is basically identical to the one presented in Chapter II. The steps of the algorithm are the same; but the definition of the costs have been changed to reflect the stochastic demand. In the deterministic case, the lowest inventory level is -1, but in the stochastic case the lowest level is -D. The costs in the disposal formula reflect this also.

This algorithm was chosen because the shape of the expected total cost function is not known. Attempts to prove that the expected total cost function is convex, or the negative of it is unimodal, have not been successful. Even establishing conditions for a local minimum have not been successful. In order to solve this problem, every inventory level between the lower and upper bound is examined. An alternative method is to select an initial inventory level, and investigate the adjacent inventory levels. To determine the minimum, all inventory levels would still have to be investigated. In general, the first level selected requires more iterations to determine the expected total cost than subsequent inventory levels require. The reason for this is that the greatest change in the estimate of the optimal cost generally occurs between the producing-for-orders-as-they-occur policy and the initial positive-inventory policy. Because the time to calculate the expected total cost depends on the number of disposal times that must be calculated, the algorithm is faster if the initial inventory policy has fewer disposal times.

The lower bound has been chosen as zero because of the complexity of computing the lower bound, and because the computed lower bound was not very restrictive for several examples. Also, because of the small number of disposal times, computing the expected total cost of low-level inventory policies does not require much computer time. This procedure yields a sensitivity analysis of the total expected cost to the inventory level. In the Implementation section of Chapter V, there is a discussion of how the sensitivity analysis can be used to assist the decision maker.

The upper bound is extremely important, because of the longer time required to calculate the expected total cost of a particular inventory level as the inventory level increases. There is an infinite possible demand, so the maximum demand is not a good upper bound. Intuitively estimating an upper bound causes two problems. First, the estimate may be larger than the algorithm's upper bound, and second, the estimate may be smaller than the optimal inventory level. The upper bound computed by the algorithm has been proven to be greater than the optimal inventory level. If there is additional information, that restricts the inventory level, such as storage space limitation, that information can easily be incorporated into the algorithm. When using the additional information as a restriction, management must be sure that the restriction actually exists. Further, management may be interested in the optimal cost if the restriction were removed. The algorithm's upper bound, even though it is complex,guarantees that the optimal inventory level can be determined. More complicated upper bounds could be constructed that would be more restrictive, but the increase in complexity is too great for the reduction in the value of the upper bound obtained. Also, a more complex upper bound would require more computer time during each revision of the bound, and would reduce the computer time savings due to decreasing the number of inventory levels that are candidates for the optimal inventory level.

Steps 1 to 9 are performed in the same manner as the deterministic case:

1) Assume the upper bound is infinite.
2) Compute $C(-1,1)$.
3) Set $m = m+1$.

4) Solve for the disposal times.

5) Revise the total cost estimate until the cost estimate converges.

6) Revise the least-cost solution if necessary.

7) Revise the upper bound.

8) If $m = m_u$, go to step 9; otherwise go to Step 2.

9) Determine the least-cost solution.

The comments concerning no solution to the disposal formula are also applicable, and have previously been discussed in this chapter. Convergence appears to be faster in the stochastic case than in the deterministic case, because the change in cost as the production level changes by one unit is less than the change for the corresponding deterministic case. This results in fewer iterations at each inventory level, because the starting point for the cost is the previous least-cost solution. In the deterministic case,that cost could be quite far from the least-cost solution for the next production level.

Revision of the upper bound is presented in detail, because it is quite different from the method in Chapter II. For each additional unit of inventory, the minimum cost per additional unit is computed and added to the cost of the present inventory level. When this cost exceeds the known least-cost solution, an upper bound has been determined. As the algorithm proceeds, both the minimum cost per additional unit,and the least-cost solution,may be revised.

By initially selecting $m_u = \infty, P_f(\infty) = 0$. This yields $C_1(m_1)$, which underestimates $C(-d_0, m_h)$. The algorithm computes the difference between cbest and $C_1(m_1)$. If this is negative, the algorithm terminates, because the underestimating function is monotone increasing. If the difference is positive, it can be used to support higher inventory levels. By computing the minimum cost per additional unit, the maximum number of additional units can be determined. This yields the new upper bound.

This procedure is illustrated in Figure 10. The dashed line indicates the actual cost for holding m_h units, and using $P_f(m_u)$ as the probability of production $C(-d_0,m_h,m_u)$. Initially $P_f(m_u) = 0$, because the upper limit is ∞. This gives $C_1(m)$. The solid line has a slope equal to the minimum cost per additional unit (mcpu), which is computed using (3.27). At the intersection of the cbest value and the solid line, a new m_u is determined.

Now the new upper bound can be substituted into (3.24). The underestimate of $C(-d_0,m)$ is larger so [cbest-the underestimate] is smaller,because $P_f(m_u)$ is larger than $P_f(\infty)$. Because mcpu is greater, and because of the higher underestimate, a new upper bound that is less than the previous m_u is determined. This revision of m_u is shown in Figure 11. Because the exact disposal times are not known for the inventory levels m_h, $m_1 \leq m_h \leq m_u$, $P_f(m_u)$ is not known exactly, but can be underestimated. By underestimating the probability of production, the maximum number of additional units is determined. The probability of future production given m units in inventory at an order point after the order has been satisfied, $P_f(m)$, is

$$(3.27)\quad P_f(m) = F(S_m) \sum_{x=1}^{D} p(x)\, P_f(m-x) + \sum_{j=1}^{m} [F(S_{m-j}) - F(S_{m-j+1})] \sum_{x=1}^{D} p(x)\, P_f(m-j-x)$$

where $P_f(k) = 1$ if $k < 0$, $P_f(0) = 1-P\infty$.

This can be underestimated by

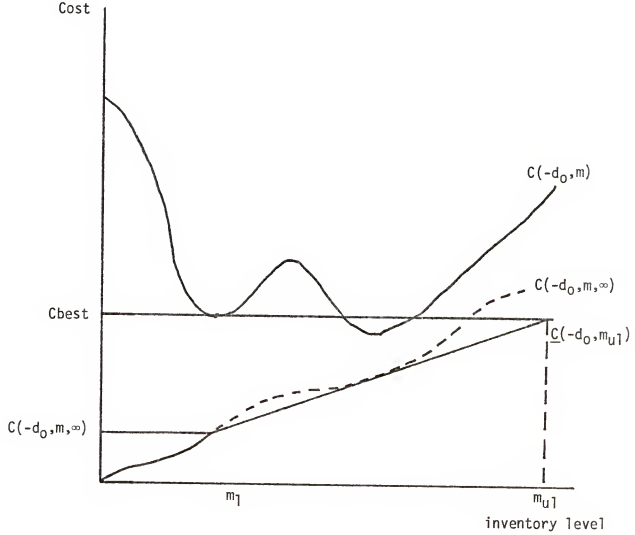


Figure 10

Determination of the Upper Bound Using mcpu and $m_u = \infty$

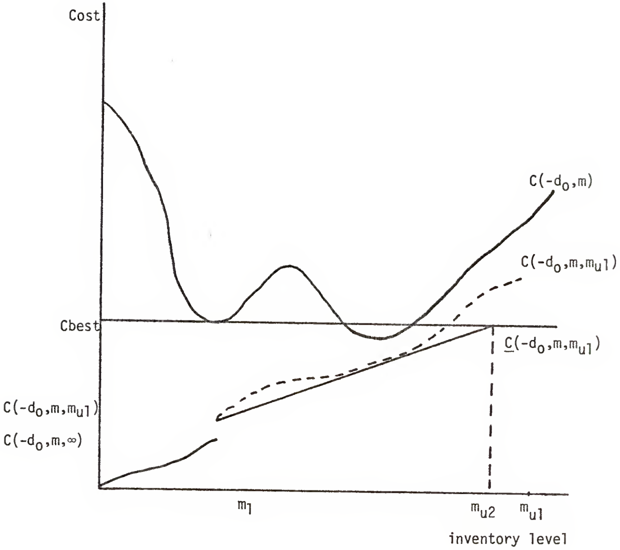


Figure 11

Determination of the Upper Bound Using mcpu and $m_u = m_{u1}$

$$(3.28)\quad \underline{P}_f(m) = F(tm) \sum_{x=1}^{D} p(x)\, \underline{P}_f(m-j) + \sum_{x=1}^{m} 0\ .$$

where $\underline{P}_f(m)$ is an underestimate of $P_f(m)$.

The minimum cost per additional unit is determined below. Each unit costs c_p to produce; or $c_p/[1-P_f(m_u)]$ is an underestimate of the expected cost, because $1/[1-P_f(m_u)]$ underestimates the expected number of production setups. This is true, because $P_f(m_u)$ underestimates the probability of production. If disposal cost is positive, an underestimate of it is $P\infty c_d$. If disposal cost is negative, the maximum probability of disposal is $[1-F(t_m)]$, so the cost recovered per unit is $[1-F(t_m)]c_d$. The appropriate cost is multiplied by the underestimate of the expected number of production setups. Holding cost is at least

$$(3.25)\quad c_h\{\int_0^{t_m} tf(t)dt + [1-F(t_m)]\ t_m\}$$

per unit, because if the unit is held, it must be held at least until the mode of the order interarrival time distribution. It is never optimal to produce units and immediately dispose of them, so in the optimal policy all units must be held at least until the mode. The minimum cost per additional unit (mcpu) is

$$(3.26)\quad \text{mcpu} = \frac{c_p + G(c_d)\ c_d + c_h\{\int_0^{t_m} t_f(t)dt + [1-F(t_m)]\ t_m\}}{1 - \underline{P}_f(m_u)}$$

$$\text{where } G(c_d) = \begin{cases} P\infty, & c_d \text{ is positive} \\ 1-F(t_m), & c_d \text{ is negative.} \end{cases}$$

The algorithm divides cbest-$C_1(m_1)$ by mcpu and truncates the result. This indicates the maximum number of additional units that can be held, and possibly have a total expected cost below cbest. A new upper bound is found by adding the truncated number of additional units to m_1, the current inventory level. The bound is not revised if the new bound is greater.

The determination of the upper bound is repeated until the truncated values are the same. If the inventory level equals the upper bound, the algorithm stops; otherwise it increases the level by one and continues the cost algorithm. Because the mcpu is positive, the algorithm requires a finite number of iterations.

Numerical Example

The example from Jesse (10) was chosen. This example was selected because Jesse's revisions to the dynamic programming algorithm have made his algorithm the fastest one for solving the stochastic order-size problem. Further, this example has many potential order sizes and a large optimal inventory level. This is important because the present algorithm is less efficient for higher inventory levels than for lower levels. Costs are given in Table 3; the conditional interarrival times are in Table 4. Fifty percent of the variable production cost can be recovered upon disposal, and the holding cost on a yearly basis is ten percent of the variable production cost. Demand is uniform on the integers one to ten, and the initial order is for five units. It is assumed that the interarrival time distribution is piecewise linear; that is, linear interpolation is used between the integers.

The cost, and times required to compute that cost, are presented in Table 5. The optimal inventory level is ten, with an associated cost of $118. Optimal disposal times are 23.4 to 23.6 months.

As the inventory level increases up to D, the disposal times become slightly longer, because the higher inventory level has a higher

TABLE 3

Data for Numerical Example

K_p = \$25.00

c_p = \$3.00

K_d = 0.

c_d = -1.50

K_h = 0.

c_h = .025 per unit per month

p_∞ = .3

D = 10

$\bar{D}$ = 5.5

TABLE 4

Conditional Interarrival Time Distribution for Numerical Example

(Months)	$\hat{f}(t)$	(Months)	$\hat{f}(t)$
1	0.0091	14	0.0548
2	0.0253	15	0.0491
3	0.0388	16	0.0430
4	0.0496	17	0.0366
5	0.0580	18	0.0302
6	0.0642	19	0.0241
7	0.0682	20	0.0181
8	0.0705	21	0.0128
9	0.0710	22	0.0081
10	0.0701	23	0.0044
11	0.0678	24	0.0017
12	0.0643	25	0.0002
13	0.0600		

TABLE 5

Results from Algorithm

Inventory Level	Cost	Computation Time	Disposal Time
1	137.01		23.4
2	136.04		23.5
3	132.75		23.5
4	130.54		23.5
5	128.21		23.5
6	125.93		23.5
7	123.79		23.6
8	121.79		23.6
9	119.94		23.6
10	118.23		23.6
11	118.37	.057 seconds	
12	118.82		
13	119.44		
14	120.16		
15	120.98		
16	121.88		
17	122.87		
18	124.25		
19	125.24	.168	
20	126.82		
21	128.32		
22	129.87		
23	131.45		
24	133.08	.389	

Time to solve problem - .389 seconds

Time to find local minimum - .057 seconds probability of satisfying the order size if there is another order. With increasing disposal times, a disposal at time S_i implies that other units with disposal times less than S_i will also be disposed. For inventory levels greater than D, the disposal times decrease.

The cost curve is insensitive around the optimum. As is characteristic of inventory models, the magnitude of the first derivative of the cost curve is in general smaller for values greater than the optimum. The computed upper bound on the optimal inventory level is 24. To calculate the optimal solution required .389 seconds.

For Jesse's dynamic programming algorithm, convergence took 19.8 seconds, or 51 times as long as the algorithm in this chapter. Finding a local minimum using the present algorithm is much faster than finding the global minimum, and in some cases may be a sufficient answer. A local minimum answer may be sufficient if the cost reduction attributed to using the local minimum solution is large,and if computer time is expensive.

The solution algorithm takes longer as the upper bound on the inventory level increases. This is because more inventory disposal times have to be computed. By reducing the possible number of order sizes, the algorithm can obtain the answer faster than for a larger number of possible order-sizes. In this example, if only five order sizes are assumed, D=5, the solution time would be only one third as long as the present time.

For a jobshop with several thousand products, the dynamic programming approach presented by Jesse would be too expensive, especially if sensitivity analysis is desired. To save computer time, management may settle for local-minimum solutions found using the solution algorithm.

Summary

Limitations on the possible candidates for the optimal production level have been established. Optimal disposal times can be determined by (3.18) for each production level. By testing all candidates, one can determine the optimal solution. Without a proof of convexity of the total expected cost formula, there is no other method of establishing the optimal solution. If the limits on inventory are close together, and if the steps in the algorithm are not too slow, the algorithm is satisfactory.

CHAPTER IV

THE FIXED COST OF DISPOSAL PROBLEM

Introduction

This chapter presents results for a variation of the problem considered in Chapter III, in which a fixed disposal cost, rather than no disposal cost, is permitted. The remaining assumptions are unchanged.

The complication introduced by this formulation is that it may be optimal to dispose of several units simultaneously, and incur only one fixed disposal cost for those units. Only two disposal policies are considered here: individual unit disposal, and disposal of all remaining inventory, hereafter called bulk disposal. Other possible disposal policies are ones in which more than one unit is disposed at some disposal point, e.g. if there are four units in inventory, the other possible disposal policies are (3,1,0), (1,3,0), (2,2,0), (2,1,1), (1,2,1), (1,1,2) where the numbers indicate how many units are disposed simultaneously at the first, second, and third disposal points, respectively.

As the inventory level increases, the number of possible disposal policies increases commensurately, because of the possible disposal combinations. To be implementable, the inventory policy chosen must not be too complex. The optimal disposal policy may depend on the inventory level, because of the fixed disposal cost and the stochastic order size, and could require a disposal policy for each inventory level, e.g. if there are four units in inventory and disposal is required, dispose two units at the first disposal point and two units at the second disposal point; if there are three or few units in inventory and disposal is required, use the bulk disposal policy. Using a different disposal policy for each level would be more complex than either of the two disposal policies: individual unit disposal and bulk disposal. The difference between the optimal cost when restricted to the two disposal policies described above, and a lower bound on the optimal cost using the optimal disposal policy,can be determined using (4.12). Unless there is a large difference, the additional expense of determining the optimal disposal policy would be greater than the expected savings. In fact, the optimal unrestricted disposal policy may be one of the two policies specified. For these reasons,and because of the complexity of other disposal policies, only the two policies above are investigated.

An upper bound on the optimal inventory level can be found by solving the problem with no fixed disposal cost. For individual unit disposal, a slight modification of the disposal time formula yields the optimal disposal times. Formulas for the bulk disposal policy are developed in this chapter. The cost of each policy is computed for each inventory level, with the lower cost policy being optimal.

In the Model Formulation section the two models are presented. The first model assumes individual unit disposal, and the second assumes that all remaining inventory is disposed of at one point in time. The individual unit disposal problem is solved by replacing

$c_d(1)$ with $K_d + c_d$ in the no-fixed-disposal-cost problem of Chapter III. In the second model, the expected total cost formula is developed, and the results are determined using a procedure analogous to the one utilized for the no-fixed-disposal-cost problem. This model is easy to construct because, for positive inventory levels, all previous inventory changes must have been caused by orders and not by disposals. The upper bound m_u from Chapter III is used, because it is an upper bound for both policies. For the deterministic order-size problem with fixed disposal costs, the results are obtained by setting $D = 1$, which is done in The Deterministic Order-Size Problem section. The solution algorithm solves three separate problems: (1) the no-fixed-disposal-cost problem, (2) the fixed-disposal-cost problem with individual unit disposal, and (3) the fixed-disposal-cost problem with all remaining inventory disposed of at one point. The first problem yields a lower bound, using (4.12), on the optimal cost of the fixed disposal cost problem, and also yields an upper bound on the optimal inventory level. The least-cost solution for the problems two and three is the optimal solution. Other disposal policies are not considered because of their complexity. The numerical example in the Numerical Example section is the same as in Chapter III, except that a fixed disposal cost is incorporated.

Model Formulation

Two different disposal policies are considered: individual unit disposal, and the all remaining unit ("bulk") disposal policies. For individual unit disposal, the results follow directly from the results of Chapter III by substituting $K_d + c_d$ for $c_d(1)$. In the bulk disposal policy, the conditioning approach used in Chapter III is again used to determine the total expected cost and disposal time formulas.

The model developed in Chapter III is extended to the fixed disposal cost, stochastic order-size problem. Holding and production cost functions are the same as in Chapter III. Disposal cost is modified to include the fixed disposal cost, where $K_d + c_d$ is the cost per disposal for the individual unit disposal policy. For the bulk disposal policy, $K_d + ic_d$ is the cost per disposal, where i is the remaining inventory level at the disposal time.

The results of Lemmas 1 and 2 of Chapter III, which are established for $K_d \geq 0$, may be used here. For the individual unit disposal policy, the cost formulas are identical to those in Chapter III, if $c_d(1)$ is replaced by $K_d + c_d$. Differentiating the total cost formula, setting the derivative equal to zero, and solving for $f(S_i)$ yields

$$(4.1) \quad f(S_i) = \frac{c_h[1 - F(S_i)]}{K_d + c_d + \sum_{x=1}^{D} p(x)\,[C(i-1-x,m) - C(i-x,m)]} .$$

The optimal disposal times are longer if there is a fixed disposal cost, because the denominator is larger. In a unimodal function in which S_i must be greater than the mode, a smaller value of $f(S_i)$ requires that S_i be larger. If the no-fixed-disposal-cost problem is solved to determine the upper bound m_u, the disposal times can be used as lower bounds on the disposal times of this problem, and possibly reduce the computations needed to determine the disposal times.

For the bulk disposal policy,there are two realizations: an order does or does not occur before disposal. These two realizations have exactly the same costs as in the individual unit disposal policy, if $m = 0$ or $m = 1$, because the two disposal policies are the same for zero or one unit in inventory. When $m \geq 2$, the two disposal policies are different. In the bulk disposal policy, if there is an inventory level i, $0 < i < m$, then there must have been orders for $m - i$ units. Disposal could not have occurred. Thus, the two types of realizations describe the order-arrival process.

For realization I the cost is

$$(4.2) \quad C_{aI}(i,m) = c_h\, i \int_0^{S_i} tf(t)dt + F(S_i) \sum_{x=1}^{D} p(x)\, C_a(i-x,m)$$

and for realization II the cost is

$$(4.3) \quad C_{aII}(i,m) = [1-F(S_i)][iS_ic_h + K_d + ic_d + C_a(-\bar{D},m)],$$ where the subscript a denotes the bulk disposal policy. These realizations are shown in Figure 12.

Adding the two formulas yields the total cost of having i units in inventory, $C_a(i,m)$. The total cost for $C_a(i,m)$ is

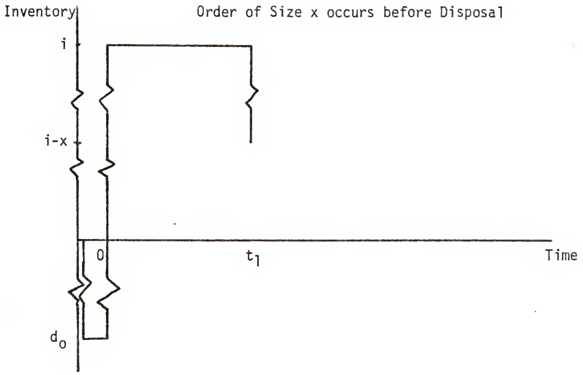
$$(4.4) \quad C_a(i,m) = c_h\, i \int_0^{S_i} tf(t)dt + F(S_i) \sum_{x=1}^{D} p(x)\, C_a(i-x,m)$$
$$+ [1-P_\infty - F(S_i)]\, C_a(-\bar{D},m)$$
$$+ [1-F(S_i)](iS_ic_h + K_d + c_d).$$

For this model the expected total cost is

$$(4.5) \quad C(-i,m) = K_p + (m+i)c_p + C_a(m,m) \qquad i = 1,2,\ldots,D.$$

To find the optimal disposal times, $C_a(m,m)$ is differentiated, the derivative is set equal to zero, and is solved for $f(S_i)$, which yields

$$(4.6) \quad f(S_i) = \frac{i\, c_h\, [1-F(S_i)]}{K_d + ic_d + C_a(-\bar{D},m) - \sum_{x=1}^{D} p(x)\, C_a(i-x,m)}\,.$$



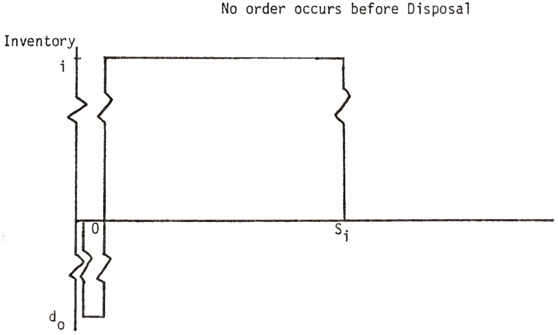


Figure 12

Realizations of the Policy to Dispose All Remaining Inventory

The total expected cost of the problem under this policy is

(4.7) $C\ (-d_0,m) = K_p + (d_0 + m)\ c_p + C_a(m,m)$

where d_0 is the size of the first order at time zero. The fixed cost of holding is considered separately.

The problem of interest is

$$(4.8) \quad C^* = \min_{m} [\ C(-i,m),\ C_a(-i,m)]$$

$$= \min_{m,S} [\ C(-i,m,S),\ C_a(-i,m,S)] \qquad i = 1,2,\ldots,D \quad S = (S_1,S_2,\ldots,S_m)$$

An upper bound for the fixed-disposal-cost problem is m_u, because the cost of holding inventory is increased by the fixed disposal costs. Also the disposal times that were determined in the no-fixed-disposal-cost problem are combined into one disposal time. If there were no fixed disposal cost, this would not be optimal. The total costs are greater for the fixed-disposal-cost problem than for the no-fixed-disposal-cost problem because of both reasons. As more disposal times are combined, the incremental amount saved by combining them diminishingly increases, so the total expected cost increases monotonically faster than in the no-fixed-disposal-cost problem. An upper bound has been established as m_u.

The Deterministic Order-Size Problem

For the deterministic order-size problem with a fixed disposal cost, the results can be obtained as a special case of the stochastic order-size problem, by requiring D = 1. In the individual unit disposal case this yields

$$(4.9) \quad f(S_i) = \frac{c_h[1 - F(S_i)]}{K_d + c_d + C(i-2,m) - C(i-1,m)}$$

as the condition for determining the optimal disposal times.

For the alternate policy, disposing of all remaining units, the total cost formula is

$$(4.10)\quad C_a(m,m) = c_h m \int_0^{S_m} tf(t)dt + F(S_m)\ C_a(m-1,m)+[1-P\infty-F(S_m)]\ C_a(\overline{D},m)$$

$$+ [1-F(S_m)](mS_m c_h + K_d + mc_d)$$

and the optimal disposal condition is

$$(4.11)\quad f(S_i) = \frac{ic_h[1-F(S_i)]}{K_d + ic_d + C_a(-\overline{D},m) - C_a(i-1,m)}$$

These are the specific formulas used in the algorithm.

The Algorithm

As part of the algorithm three problems are solved: (1) the no-fixed-disposal-cost problem, (2) the fixed-disposal-cost problem with individual unit disposal, and (3) the fixed-disposal-cost problem with bulk disposal. The optimal solution from the no-fixed-disposal-cost problem is used as an upper bound for the fixed-disposal-cost problem. By computing the expected cost of each inventory level for both policies, the optimal solution is obtained. Because only two disposal policies are considered, there may be a more complex policy that has a lower expected cost. A lower bound on the cost of the optimal policy,regardless of disposal policy, is determined using (4.12) from the no-fixed-disposal-cost problem. If the least-cost solution obtained from the two disposal policies is close to the lower bound on the optimal cost regardless of policy, then the solution is near optimal. The small potential savings would not justify the additional computational expense. Even if the potential savings are large, the complexity of the disposal policy necessary to obtain the savings may be so great that management may not want to use the policy. Management has to define an acceptable level for the potential savings, and must consider the calculation cost, the difficulty of using the policy, the validity of the model, and the accuracy of the data when determining the acceptable level.

The algorithm is similar to the one used in Chapter III except that it is applied to problems one, two, and three sequentially, and then the minimum of the solutions to problems two and three is determined. First, the algorithm solves the no-fixed-disposal-cost problem. By adding the fixed disposal cost multiplied by the probability of a disposal to $C(d_0,m^*)$, a lower bound on the fixed-disposal-cost problem cost is computed,

$$(4.12) \quad C_a(-1,m^*) \geq C(-1,m^*) + K_d\{P\infty \cdot 1/[1-P_f(m^*)]\}$$

where the underestimate of the expected number of production setups is calculated by

$1 / [1 - P_f(m^*)]$.

This is an underestimate, because the m^* in problem one is an upper bound on the optimal inventory level for the fixed-cost-of-disposal problem, and because $P_f(m)$ is monotone decreasing. The probability of at least one disposal per production setup is $P\infty$. Next the algorithm solves the problem with the fixed disposal cost and individual disposals. The total cost is computed for the bulk disposal policy in the same manner as for the individual unit disposal policy, only the cost and disposal formulas are different.

There are now three costs computed for the problem: the cost with no fixed disposal cost plus an underestimate of the disposal cost which is the right hand side of (4.12), the cost for individual disposal, and the cost for bulk disposal. From the last two, the lower cost is the optimal cost. Management should compare this to the RHS of (4.12). If the difference is small, the policy chosen is very close to the optimal policy for the problem that permits any type of disposal policy. For small differences, management is justified in limiting disposal policies to those which are easily implemented. For a large difference, management needs to consider if this difference justifies considering more complex disposal policies. Investigation of other disposal policies may also indicate that the previously found policy is optimal.

Numerical Example

In this section both the optimal policy, regardless of disposal policy, and the optimal solution restricted to the two disposal policies, individual unit and bulk disposal, are compared. The differences in the computational time required to obtain the solutions are compared. For the example given, the effect of the fixed disposal cost is shown.

The same example used in Chapter III is again used in this section. Now, however, the fixed disposal cost is changed to a positive value. The problem data are presented in Table 6. Fifty percent of the variable production cost can be recovered upon disposal, and the holding cost on a yearly basis is ten percent of the variable production cost. Demand is uniform on the integers one to ten, and the initial order-size is five units. It is assumed that the interarrival time distribution is piece-wise linear; that is, linear interpolation is used between the integers.

TABLE 6

Problem Data for Fixed-Disposal-Cost Example

K_p = \$25.00

K_d = 5.00

K_h = 0.

P_∞ = .3

c_p = \$ 3.00 per unit

c_d = -1.50 per unit

c_h = .025 per unit month

$\bar{D}$ = 5.5

D = 10

The optimal solution is to produce enough units to make the inventory level ten, after satisfying the current order,when production is required. If the first order is for five units, the optimal cost is \$118.48. The disposal time is 23 for units 2-10 and 24 for unit 1. This solution was determined by dynamic programming, and required 22.4 seconds of computer time.

For the individual item disposal policy, the optimal solution is to hold zero inventory. This solution has an expected cost of \$137, whereas holding ten units has an expected cost of \$152. In the bulk disposal policy, the optimal policy is to hold ten units. This policy has a cost of \$123. The disposal time depends on the number of units in inventory at the time of disposal. The times are given below:

Inventory level	0	1	2	3	4	5	6	7	8	9	10
Disposal time	-	24.5	24.2	24.0	23.9	23.9	23.8	23.8	23.8	23.7	23.7

Total time to solve this problem on the computer is .50 seconds, or 45 times faster than dynamic programming. The difference in the optimal costs between the two algorithms is due to several factors:

1. The dynamic program assumes only 12 potential reorders; the cost increases by 40¢ if 12 instead of 11 future orders are allowed. Increasing the number of potential orders to 20 or 25 would increase the cost somewhat, but would double the computer time required to solve the problem.
2. In the present algorithm a convergence value of 50¢ was used. Smaller values would decrease the cost, because the algorithm approaches the total cost from above.
3. Only two disposal policies are considered in the algorithm presented in this chapter.

With several thousand products in a jobshop, using dynamic programming to solve those problems would take a considerable amount of time. Furthermore, the computational cost is greater and could exceed the potential savings.

The insensitivity of the solution to changes in the fixed disposal cost is revealed in Table 7. For the fixed disposal cost of 5¢ or more, the disposal of all remaining units is a better policy than individual unit disposal.

The present algorithm yields a solution that is close to the optimal solution, given an unrestricted disposal policy. Additional computational costs are considerable, and may exceed potential savings. This savings is also offset by having to use a more complex disposal policy.

TABLE 7

Fixed Disposal Cost Effects on the Optimal Policy

OPTIMAL POLICY

K_d	Individual Disposal		Disposal of All Units	
	Cost	Inventory Level	Cost	Inventory Level
.05	$ 119	10	$ 118	10
1.00	125	10	119	10
3.00	137	0	121	10
5.00	137	0	123	10
10.00	137	0	128	10

CHAPTER V

CONCLUSIONS

Summary of Major Contributions Made by this Dissertation

This dissertation presents results that limit the candidates for the optimal inventory policy in two types of problems: (1) deterministic order size, uncertain future order occurrence, and (2) stochastic order size, uncertain future order occurrence. Each of these two types of problems has been divided into two subproblems: the no-fixed-disposal-cost and the fixed-disposal-cost subproblems. In particular, for the fixed-disposal-cost subproblems, conditions are presented for the optimal disposal times for two types of disposal policies, individual unit and bulk disposal, as well as conditions that set an upper bound on the optimal inventory level. Only the individual unit disposal policy is presented for the no-fixed-disposal-cost subproblems, because that policy is shown to be optimal. These conditions permit the use of a more efficient algorithm than dynamic programming, the previous solution method.

Chapters II and III discuss the no-fixed-disposal-cost subproblems. These subproblems are examined assuming linear variable costs of holding, production, and disposal. Fixed production and holding costs are also permitted.

The optimal disposal policy for the no-fixed-disposal-cost problem is shown to be individual unit disposal. Using an iterative algorithm, the optimal cost and policy disposal times are determined. In addition, the expected costs for holding smaller integer inventory levels are also computed. This sensitivity analysis could be useful in multiproduct problems and is discussed in the Implementation section. The algorithm is hundreds of times faster than dynamic programming, averaging 800 times faster in the deterministic case over 31 test problems. This great increase in speed makes solving thousands of job-shop problems economically feasible. Management can request an extensive sensitivity analysis, and not fear the computational expense. Although the algorithm is not as fast in the stochastic case, it is 50 or more times faster than dynamic programming. The algorithm's speed is sensitive to the number of possible order sizes. The example problem was selected because it had ten units as the optimal inventory level, which is large enough for many jobshop problems, particularly in the printing industry.

For the fixed-disposal-cost subproblems, discussed in Chapter IV, the algorithm computes the expected cost and disposal times of two disposal policies: individual unit disposal and bulk disposal. It is observed that bulk disposal is less expensive if the disposal cost is high compared to the other expected costs measured at the disposal point. While this policy may not always be optimal, it is shown that it is a good solution. The "goodness" of this policy can be tested by comparison to a lower bound. The "goodness" may be great enough that further computation is not economically justified. Further, management often wants to use simple disposal policies examined in this dissertation.

The theoretical results reduced the computations necessary,so that an algorithm could be developed that could solve the problems inexpensively. Firms in the printing industry, which have a large number of products, cannot afford to spend the money necessary to solve this problem for each product by dynamic programming. Performing sensitivity analysis by dynamic programming is prohibitively expensive. Although substantial savings can be obtained by holding inventory for problems with relatively large setup costs, only two firms of ten surveyed by the author produce for inventory, and neither of these firms knows the optimal inventory levels. There has been great reluctance to hold inventory, especially if it is expensive to determine how much inventory to hold. In this dissertation,the cost of computing the optimal inventory level has been substantially reduced.

The problems investigated in this dissertation are important because several industries, such as printing, electronics, automobile replacement parts, fashion clothing, and government contracting, have this type of problem. The number of products with uncertain order-occurrence is substantial,because of the large number of firms in those industries,and because of the number of products in each firm. Further, in order to solve the more complicated uncertain order-occurrence problems that have shortage costs, multiple products, finite production rates, or changing parameters, results for the uncertain order-occurrence problems in this dissertation need to have been previously established. Those results can serve as a beginning for investigation of the more complex order-occurrence problems.

Implementation

To implement this method several restrictions need to be considered. Will disposal result in sanctions against the manager? Is there sufficient capital, warehouse space, and production capacity to implement this method? The potential sanctions against the manager could cause him/her not to adopt the present method, even though there is an expected cost reduction. By producing less than the optimal quantity for inventory, the probability of disposal is decreased, perhaps to an acceptable level. Also the manager could compare wasting manhours setting up the machinery, to wasted materials and production time. In industries such as printing, the setup time is considerable, and the manager should be looking for ways to decrease it. Also, the manager could select a few items to initially produce for inventory in order to reduce the probability of disposal occurring.

To handle the capital, space, and production constraints, first management should compute the optimal inventory levels for each product, and the costs for less than optimal production levels, and then determine the total capital, space, and production requirements. If these requirements are less than the available resources, management should proceed in implementing the optimal inventory levels. If an easy to-determine schedule can be found for producing the optimal quantities, even though the requirements are greater than the available resources, then that schedule should be used. If this cannot be done, then the most restrictive constraint should be determined. If it is a capital constraint, then capital should be obtained, either items that require small capital expenditures, should be produced, or less than optimal quantities should be produced. Even producing less than the optimal quantities can result in substantial savings. These savings can be used to supply capital for the other items. For a shortage of storage space, either more space should be acquired by renting or salvaging usable space in the plant, or items that do not require much space individually should be produced or less than the optimal quantities should be produced. A similar procedure should be followed for a shortage of production capacity. Overtime or subcontracting other jobs could be done temporarily until the initial production requirements are fulfilled. Another alternative is to choose an item that will likely be ordered soon and has a long setup time, but a short production time. Less than the optimal quantity can be produced, but this quantity must be a multiple of the order-size for best results. This procedure will make available additional production time as soon as possible. The easiest time to start implementing the policy is during a slack time when there is idle production capacity.

In some industries such as printing, there is a correlation between items that require small amounts of capital, space, and production time. This eases the selection of production levels, and items to produce. One method of meeting the constraints is to equally apportion the resources among products. This is not optimal, but is easy to implement. Unfortunately, some orders may not be filled by this method. Another method is to have three to five classes of products, e.g. the ABC inventory method. The most important items in terms of setup time, or setup time divided by production time, would be in class A. Class A items would be the first items produced in quantities larger than the order-size. Depending on resource availability, the class A items could be produced at optimal, or near optimal levels, before other items were produced.

Using the sensitivity analysis, the manager could determine how much of each item to produce in the multiproduct problem. This could be done by putting the incremental savings associated with each unit increase in production for each item in descending order. The highest items on the list, the unit production increases resulting in the most savings, could be implemented first. In case of ties or near ties, the capital, space, or production time requirements could be considered when the decision is made.

The model is a simplification of reality in several ways: parameters are assumed constant, production time, shortage costs, inventory deterioration, and allocation of resources between items are not in the model. These results are important, because there are industries, such as the printing industry, that have this problem, and do not know how to solve it. Improvement that an approximate solution yields can be quite significant.

The shortcomings mentioned are not as severe as they appear to be. Because the algorithm is fast and relatively inexpensive, optimal inventory and production levels can be recomputed,if desired,just before production is started. This will decrease the error due to changes in the parameters since the previous production point. Scheduling difficulties may increase as the method is implemented, but then they will decrease, as more time will be available from the setup time savings. Ways to minimize the production time requirements were previously discussed in this section.

Shortage costs can be important. Customers are used to waiting for the special orders, but not for excessive amounts of time. By having more usable capacity due to decreased setup time requirements, the wait can be reduced, sometimes even eliminated. This reduction can be a major selling point for repeat business. By increasing repeat business, the firm can reduce uncertainty related to losing the job to another firm. Handling the shortage costs is extremely complex, because shortage costs are neither strictly time costs, nor unit costs. The model presented should be exploited before such complications are introduced.

Inventory deterioration can be a major problem. If adequate care and storage techniques are used, this could be much less of a problem. For example in the printing industry, dust and humidity could make inventory unusable. By storages in boxes in air conditioned space, the loss would be minimal. The cost of these techniques is included in the holding cost. Other industries may have similarly easy to implement methods of avoiding deterioration, loss, or theft.

Allocation of resources between items for inventory has always been a complex problem. The techniques described in the implementation paragraphs could be used on a continuing basis. This method may result in a suboptimal overall policy, but there is evidence from the sensitivity analysis that near optimal costs can be achieved over a range of policies. This is not surprising, and is similar to other inventory models, such as the EOQ model.

There are several simplifications of reality that are used in order to make the model simple enough to solve. The results from this model can result in large improvements in the operation of the jobshop, and the cost of its operation. Further investigation into relaxing each of the assumptions are possible future research areas.

Recommendations for Future Research

Several theoretical areas are available for extending this work. Extensions will not be easy because of the complexity of the problems, and the difficulty in proving convexity and monotonicity of the costs. Assuming simpler cost function may make the task easier, but yields a less realistic model. Discounting of costs could be studied. Revising the upper bound on the optimal inventory level is another useful extension. Proof of convexity of the total expected cost formula would greatly help, especially in establishing bounds on the inventory level. Further investigation into the optimal disposal policy types for the fixed-disposal-cost problem is another area. Lastly, a multiproduct problem could be attempted. Historically these have been the most difficult to solve, and in this case the same appears true. Obtaining these theoretical results will be extremely time consuming.

One method of solving the uncertain order-occurrence problem is to formulate it as a semi-Markov process. An absorbing state represents the termination of the order process. Two states are defined for each nonnegative inventory level less than the maximum inventory level: one for each type of transition, i.e. order occurrence or disposal, that can occur to reach that level of inventory. This formulation assumes that it can be shown that the disposal times are independent of the process realization. If the disposal times cannot be shown to be independent, then more states are necessary. Using the method in this dissertation, the disposal times have been shown to be independent. Associated with each state are the costs of transition into that state from the other states. The cost of transition is the expected holding and disposal costs for nonproduction states and the cost of production for the producing states.

The semi-Markov formulation has several, severe disadvantages. A large transition matrix is required because the number of states grows as either the inventory level, or the possible number of order sizes, D, increases. For the example in Chapter III, with an inventory level of ten units, ten possible order sizes, and individual unit disposal, there are 32 states if the disposal times can be shown to be independent, and 77 states if the disposal times cannot be shown to be independent. The fundamental matrices have 270 and 801 nonzero entries for the 32 and 77 state models, respectively.

Another disadvantage is that the transition probabilities are not known because the optimal disposal times are not known. An iterative procedure for estimating the disposal times, computing the fundamental matrix, and then reestimating the disposal times could be used, but would require several iterations. Estimating disposal times is necessary because a noniterative condition for determining disposal times has not been devised. At each inventory level that is a candidate for the optimal inventory level, another fundamental matrix must be computed. If there are many candidates, this algorithm is time consuming and expensive. In order to have few candidates, very restrictive bounds are needed. Determining such restrictive bounds could be very difficult. The severe disadvantages of the semi-Markov formulation indicate a low probability of success in the author's opinion.

In contrast to the difficulty of obtaining theoretical results, there are several applied extensions that would not be too difficult.

An interesting area is the actual application of this algorithm to a jobshop, such as a printer. The determination of the costs, order size, probability of a reorder, and the reorder rate function, especially determining if the values are constant over time, would be interesting. Tables or heuristics showing the optimal policy under various conditions could be developed for management. Particularly important are heuristics that describe the effects of the reorder probability, fixed production cost, and the holding cost. The effect of using a smaller value of D is also a good area in which to work, because the computational effort needed to solve a problem can be reduced by using smaller values of D. A faster algorithm could be developed using the approximation of $1-P_\infty$ for $F(S_i)$. Further investigation would be necessary to determine when this is a good approximation. There are several areas for both the theoretical and applied extension of the work presented in this dissertation.

APPENDIX A

DERIVATION OF THE OPTIMAL DISPOSAL TIME EQUATION

The derivative of (2.16) is

$$\frac{\partial C(-1,m)}{\partial S_1} = \frac{\partial}{\partial S_1} \{K_p + mc_p + \sum_{k=1}^{m} [F(S_{k-1}) - F(S_k)][(\sum_{L=k+1}^{m-1} (c_d + c_h S_L)$$

$$+ C(k-2,m)] + (k-1)c_h \int_{S_k}^{S_{k-1}} tf(t)\,dt\}$$

$$= \frac{\partial}{\partial S_1} \{ K_p + mc_p + [1-F(S_1)][\sum_{L=2}^{m-1} (c_d + c_h S_L) + C(-1,m)]$$

$$+ 0 + \int_{S_2}^{S_1} f(t)dt[\sum_{L=2}^{m-1} (c_d + c_h S_L) + C(0,m)]$$

$$+ \sum_{k=3}^{m} [F(S_{k-1}) - F(S_k)]C(k-2,m) + c_h \int_{S_2}^{S_1} +f(t)\,dt\}$$

$$= -f(S_1) [\sum_{L=1}^{m-1} (c_d + c_h S_L) + C(-1,m)] + 0 + f(S_1)[\sum_{L=2}^{m-1} (c_d + c_h S_L)$$

$$+ C(0,m)] + c_h[1-F(S_1)] + \sum_{k=3}^{m} \{[F(S_{k-1}) - F(S_k)]\frac{\partial C(k-2,m)}{\partial S_1}\}$$

$$+ c_h S_1 f(S_1)$$

where $S_o = \infty$, $S_m = 0$

$$= -f(S_1) [c_d + c_h S_1 - C(0,m) + C(-1,m)] + c_h S_1 f(S_1)$$

$$+ c_h [1-F(S_1)] + \sum_{k=3}^{m} [F(S_{k-1}) - F(S_k)] \frac{\partial C(k-2,m)}{\partial S_1}$$

For all values of $k \geq 3$, we have $C(k-2,m) = \alpha_k + \beta_{1k}\, C(-1,m)$.

$$\frac{\partial C(-1,m)}{\partial S_1} = -f(S_1)\ [C(-1,m) + c_d - C(0,m)] + c_h\ [1-F(S_1)]$$

$$+ \sum_{k=3}^{m} [F(S_{k-1}) - F(S_k)]\ \beta_{1k}\ \frac{\partial C(-1,m)}{\partial S_1}\ dt$$

$$= \{1 + \sum_{k=3}^{m} [F(S_{k-1}) - F(S_k)]\beta_{1k}\}\{-f(S_1)[c_d + C(-1,m) - C(0,m)]$$

$$+ c_h[1 - F(S_1)]\}$$

or

$$f(S_1) = \frac{c_h[1 - F(S_1)]}{c_d + C(-1,m) - C(0,m)}$$

Similarly for S_2 we have

$$f(S_2) = \frac{c_h[1 - F(S_2)]}{c_d + C(0,m) - C(1,m)}$$

For general values of $i < m$

$$\frac{\partial C(-1,m)}{\partial S_i} = \frac{\partial}{\partial S_i}\ \{\ K_p + mc_p + \sum_{k=1}^{m-1} [F(S_{k-1}) - F(S_k)]$$

$$[\sum_{L=k}^{m-1} (c_d + c_h S_L) + C(k-2,m)]$$

$$+ (k-1)c_h \int_{S_k}^{S_{k-1}} t\ f(t)\ dt]$$

$$+ [F(S_{i-1}) - F(S_i)]\ [\sum_{L=i}^{m-1} (c_d + c_h S_L) + C(i-2,m)]$$

$$+ (i-1)c_h \int_{S_i}^{S_{i-1}} t\ f(t)\ dt$$

$$+ [F(S_i) - F(S_{i+1})] [\sum_{L=i}^{m-1} (c_d+c_hS_L) + C(i-2,m)]$$

$$+ i\ c_h \int_{S_{i+1}}^{S_i} t\ f(t)\ dt$$

$$+ \sum_{k=i+2}^{m-1} \{[F(S_{k-1}) - F(S_k)][\sum_{L=k}^{m-1} (c_d+c_hS_L) + C(k-2,m)]$$

$$+ (k-1)\ c_h \int_{S_k}^{S_{k-1}} t\ f(t)\ dt]\}$$

$$= \sum_{k=1}^{i=1} [F(S_{k-1}) - F(S_k)]\ c_h - f(S_i)[\sum_{L=i}^{m-1} (c_d+c_hS_L) + C(i-2,m)]$$

$$- (i-1)\ c_hS_if(S_i) + c_h\ [F(S_{i-1}) - F(S_i)] + ic_hS_i\ f(S_i)$$

$$+ f(S_i)\ [\sum_{L=i+1}^{m-1} (c_d+S_Lc_h) + C(i-1,m)]$$

$$+ \sum_{k=i+2}^{m-1} [F(S_{k-1}) - F(S_k)]\ \frac{\partial C(k-2,m)}{\partial S_i}$$

Recall that $\frac{\partial C(k-2,m)}{\partial S_i} = \beta_{ik} \frac{\partial C(1-,m)}{\partial S_i}$ for $k-2 > i$

$$\frac{\partial C(-1,m)}{\partial S_i} = \{1 + \sum_{k=i+2}^{m-1} [F(S_{k-1}) - F(S_k)]\beta_{ik}\}$$

$$\{\ c_h[1 - F(S_i)] - f(S_i)\ [c_d+C(i-2,m)-C(i-1,m)]\}$$

Setting the derivative equal to zero yields

$$f(S_i) = \frac{c_h[1 - F(S_i)]}{c_d+C(i-2,m) - C(i-1,m)}\ .$$

APPENDIX B

PROOFS OF THEOREMS 1,2,3,4

Theorem 1. The immediate production cost component is convex for all positive production levels.

It is a linear cost function for positive production, and, hence, convex for $m > 0$.

Theorem 2. Holding costs between the current production point and the next production point are monotone increasing in m.

Proof: We examine (2.15) for m and m + 1. Define $C_h(k,j)$ as the holding cost component of $C(k,j)$.

$$C_h(m,m+1) = \sum_{k=0}^{m} \{kc_h \int_{S_{k+1}}^{S_k} tf(t)dt + [F(S_k)-F(S_{k+1})][C(k-1,m+1)$$

$$+ c_h \sum_{n=k+1}^{m} S_n]\} + P\infty(c_h \sum_{n=1}^{m} S_n)$$

$$C_h(m-1,m) = \sum_{k=0}^{m-1} \{kc_h \int_{S_{k+1}}^{S_k} tf(t)dt + [F(S_k)][C(k-1,m) + c_h \sum_{n=k=1}^{m-1} S_n]\}$$

$$+ P\infty(c_h \sum_{n=1}^{m-1} S_n)$$

Realizing that $C_h(i,m+1) = C_h(i,m)$, we have

$C_h(m,m+1) + C_h m-1,m) =$

$$= mc_h \int_0^{S_m} f(t)dt + F(S_m)\ C_h(m-1,m+1) + (m-1)c_h \int_{S_m}^{S_{m-1}} tf(t)dt$$

$$+ F(S_{m-1}) - F(S_m)\ [C_h(m-2,m+1) - c_h S_m] + P\infty C_h S_m$$

$$- [(m-1)c_h \int_0^{S_{m-1}} tf(t)dt + F(S_{m-1})\ C(m-2,m)]$$

$$= c_h \int_0^{S_m} tf(t)dt + [F(S_{m-1}) - F(S_m)]\ (c_h S_m) - P\infty c_h S_m$$

$$+ F(S_m)[C_h(m-1,m+1) - C_h(m-2,m)]$$

or

$$[1-P\infty - F(S_m)][C_h(m,m+1) - C_h(m-1,m)]$$

$$= c_h \int_0^{S_m} tf(t)dt + P\infty c_h S_m + [(F(S_{m-1}) - F(S_m)]\ (c_h S_m)$$

which is positive if $S_m \neq 0$ and zero if $S_m = 0$, hence $C_h(m,m+1) - C_h(m-1,m) \geq 0$ and the theorem is proved.

Theorem 3. The expected disposal cost between the initial production point and the next production point is monotone increasing in m.

Proof: Following the steps in the proof of Theorem 2, we define $C_d(k,j)$ as the disposal cost component of $C(k,j)$.

$$C_d(m,m+1) = \sum_{k=0}^{m} \{[F(S_k) - F(S_{k-1})][C_d(d-1,m+1) + (m-k)c_d(1)]\} + P\infty mc_d(1)$$

$$C_d(m-1,m) = \sum_{k=0}^{m-1} \{[F(S_k) - F(S_{k-1})][C_d(k-1,m) + (m-1-k)c_d(1)]$$

$$+ P\infty(m-1)c_d(1)$$

then

$$C_d(m,m+1)-C_d(m-1,m+1) = F(S_m)[C_d(m-1,m)] + P\infty c_d(1)$$

$$+ [F(S_{m-1})-F(S_m)][C_d(m-2,m)+c_d(1)]-F(S_{m-1})C_d(m-2,m)$$

$$=F(S_m)[C_d(m-1,m+1)-C_d(m-2,m)] + P\infty c_d(1) +[F(S_{m-1})-F(S_m)]c_d(1)$$

or

$$[1-F(S_m)][C_d(m,m+1)-C_d(m-1,m)] = P\infty c_d(1)+[F(S_{m-1})-F(S_m)]c_d(1)$$

which is positive, hence the theorem is proved.

Theorem 4. The future cost component is convex monotone decreasing in the production level.

proof: Examine $P_f(1)$, $P_f(2)$, $P_f(3)$. Define $F(S_j) = \int_0^{S_j} f(t)dt$.

$$P_f(1) = 1 - P\infty$$

$$P_f(2) = P_f(1)\ F(S_1) + (1-P\infty)-F(S_1) = 1 - P\infty-P\infty F(S_1)$$

$$P_f(3) = P_f(2)F(S_2) + [F(S_1)-F(S_2)]P_f(1) + 1-P\infty-F(S_1)$$

$$= 1-P\infty-P\infty F(S_1)-P\infty F(S_1)F(S_2)$$

In general

$$P_f(1) = 1 - P\infty$$

$$P_f(m) = 1 - P\infty-P\infty F(S_1)-P\infty F(S_1)F(S_2)-\ldots-P\infty F(S_1)\ldots F(S_{m-1})$$

$$= 1 - P\infty-P\infty\sum_{k=1}^{m-1} \left(\prod_{i=1}^{k} F(S_i)\right) \quad m = 2,3,\ldots$$

For each m define

$$\Delta P_f(m) = P_f(m+1) - P_f(m) = -P_\infty \prod_{i=1}^{m} F(S_i)$$

Since $F(t) \geq 0$, for $t \geq 0$, and $P_\infty \geq 0$, $\Delta P_f(m)$ is decreasing in m.

Further,

$$\Delta P_f(m+1) - \Delta P_f(m) = -P_\infty \left(\prod_{i=1}^{m+1} F(S_j) - \prod_{i=1}^{m} F(S_i) \right)$$

$$= +P_\infty [1 - F(S_{m+1})] [\prod_{i=1}^{m} F(S_i)]$$

The above shows that the increment becomes smaller as m increases because ΔP_c (.) are negative. This is sufficient to prove that $P_f(m)$ is convex monotone decreasing.

APPENDIX C

DISCUSSION OF INTEGER INVENTORY ASSUMPTION

This assumption is not very restrictive because the inventory level must be greater than or equal to -D because of the no backlogging assumption, and because the assumption has been proved for the rational numbers. The proof follows.

From Jesse (10) Theorems 3.4 and 3.5, we know that the optimal inventory level is the sum of the next n potential order sizes, n=0,1,2,... . Because the order sizes are integers, the optimal inventory level is integer valued. Jesse's proof requires a countable number of points at which the inventory level can change. These points are potential disposal times. The rational numbers are countable, so at any rational number the optimal inventory level is integer valued.

Knowing the optimal inventory level at the rational numbers is integer valued is sufficient for almost all applications, because any real number can be approximated to any desired accuracy by a member of the rational numbers, and because almost all firms specify policies in terms of rational numbers. In order to take derivatives, it is necessary to assume the optimal inventory level at all irrational numbers is integer valued.

APPENDIX D

ANALYTICAL SOLUTION FOR C(1,1)

To determine the analytical solution for C(1,1), substitute for C(0,1) in (3.7) and combine terms.

$$C(1,1)= \sum_{x=2}^{D} P(x) \quad c_h\int_0^{S_1} tf(t)dt + \quad K_p+xc_p + C(1,1) \quad F(S_1)$$

$$+ P(1) \quad c_h\int_0^{S_1} tf(t)dt + (1-P_\infty) \, [K_p+(\overline{D}+1)c_p + C(1,1) \, F(S_1)]$$

$$+ [1-F(S_1)][(c_hS_1 + c_d(1)]$$

$$+ [-P_\infty+\overline{F}(S_1)] \sum_{x=1}^{D} p(x)[K_p + c_p(x+1) + C(1,1)]$$

$$= c_h\int_0^{S_1} tf(t)dt + \overline{F}(S_1) \, [c_hS_1 + c_d(1)]$$

$$+ [K_p + C(1,1)][F(S_1) - P_\infty P(1)F(S_1) - \quad P_\infty + \overline{F}(S_1)]$$

$$+ c_p\{\overline{F}(S_1) - P_\infty - P_\infty F(S_1)p(1) + \overline{D}[1-P_\infty + (1-P_\infty)F(S_1)p(1)]\}$$

$$= A + B + [K_p + C(1,1)] \, C + E$$ where the letters correspond to the terms in order

or

$$C(1,1) = \frac{A + B + K_p C + E}{1 - C}$$ which is a complex formula.

APPENDIX E

DERIVATION OF THE OPTIMAL DISPOSAL TIMES

The procedure used is to differentiate the total expected cost formula with respect to the disposal time in question, set the derivative equal to zero, and combine terms to form the necessary condition. The derivative is

$$\frac{\partial C(i,m)}{\partial S_i} = \frac{\partial}{\partial S_i}\{\sum_{k=0}^{i=1}\{kc_h \int_{k+1}^{S_k} tf(t)dt + [1 - F(S_{k+1})]S_{k+1}c_h$$

$$+ ic_h \int_0^{S_i} tf(t)dt + \sum_{k=1}^{i}[1-F(S_k)]c_d\ (1)$$

$$+ \sum_{k=1}^{i+1}\ \sum_{x=1}^{D}\ [p(x)\ C(k-1-x,m)]\ [F(S_{k-1}) - F(S_k)]\}$$

Apply the rule for differentiating integrals.

$$= \underset{\text{from the integral}}{\underline{-(i-1)c_h S_i f(S_i)}} + \underset{\text{from the second term in the sum.}}{\underline{[1 - F(S_i)]c_h - f(S_i)S_i c_h}}$$

$$+ \underset{\text{From "m" integral}}{\underline{ic_h S_i f(S_i)}} \quad \underset{\text{from } c_d \text{ term}}{\underline{-\ f(S_i)c_d(1)}}$$

$$+ \underset{\text{when}\quad i=k-1}{\sum_{x=1}^{D}\ [p(x)\ C(i-x,m)]\ f(S_i)} - \underset{\text{when}\quad i=k}{\sum_{x=1}^{D}[p(x)\ C(i-1-x,m)]\ f(S_i)}$$

By combining terms the result is

$$= f(S_i)\{-c_d(1) + \sum_{x=1}^{D} p(x)[C(1-x,m) - C(i-1-x,m)]\} +$$

$$+ [1-F(S_i)]c_h.$$

Setting the derivative equal to zero and moving the $f(S_i)$ term to the other side gives the desired result

$$f(S_i) = \frac{[1-F(S_i)]c_h}{c_d(1) + \sum_{x=1}^{D} p(x)[C(i-1-x,m) - C(i-x,m)]} .$$

APPENDIX F

PROOFS FOR THEOREMS 1, 2, 3, 4

Theorem 1. The current production cost is convex monotone increasing for all positive production levels.

Proof: The initial production cost is convex for all positive production levels. This is true because the initial production cost changes by a linear amount over changes of positive production. Linear functions are convex.

Theorem 2. Holding costs between one production point and the next production point are monotone increasing in m.

Proof: Holding costs between the initial production point and the next production point are monotone increasing as m increases, if a specified disposal policy is used, e.g. individual unit disposal. Examine $C_h(m,m)-C_h(m-1,m) = \Delta C_h(m-1,m-1)$, where $C_h(m,m)$ is the expected holding cost of $C(m,m)$.

$$\Delta C_h(m-1,m-1), = \sum_{k=1}^{m} kc_h \int_{S_{k+1}}^{S_k} tf(t)dt - \sum_{k=1}^{m-1} kc_h \int_{S_{k+1}}^{S_k} tf(t)dt$$

$$+ \sum_{k=1}^{m} [1-F(S_k)]S_k c_h - \sum_{k=1}^{m-1} [1-F(S_k)]S_k c_h$$

$$+ \sum_{k=1}^{m+1} \sum_{x=1}^{D} p(x)C_h(k-1-x,m)[F(S_{k-1}) - F(S_k)]$$

$$- \sum_{k=1}^{m} \sum_{x=1}^{D} p(x)C_h(k-1-x,m-1)[F(S_{k-1}) - F(S_k)]$$

Because only the production level changes, $C_h(k-1-x,m) = C_h(k-1-x,m-1)$. This simplifies to

$$\Delta C(m-1,m-1) = mc_h\int_0^{S_m} tf(t)dt + (m-1)c_h\int_{S_m}^{S_{m-1}} tf(t)dt$$

$$-(m-1)c_h\int_0^{S_{m-1}} tf(t)dt + [1-F(S_m)]S_m c_h$$

$$+ \sum_{x=1}^{D} p(x)C_h(m-x,m)F(S_m) + \sum_{x=1}^{D} p(x)C_h(m-1-x,m)[F(S_{m-1})-F(S_m)]$$

$$- \sum_{x=1}^{D} p(x)C_h(m-1-x,m-1)\ F(S_{m-1})$$

or

$$\Delta C(m-1,m-1) = \frac{c_h\int_0^{S_m} tf(t)dt + [1-F(S_m)]\ S_m c_h}{1 - F(S_m)}$$

which is nonnegative and thus the theorem is proved.

Theorem 3. The expected disposal cost between the initial production point and the next production point is monotone increasing in the production level if $c_d \geq 0$. If $c_d < 0$ then unit production plus unit disposal costs is monotone increasing.

Proof: $c_d \geq 0$. The steps are similar to the steps in the proof of Theorem 2.

$$\Delta C_d(m-1,m-1) = C_d(m,m) - C_d(m-1,m-1)$$

$$= \sum_{k=1}^{m} [1-F(S_k)]c_d(1) + \sum_{k=1}^{m} \sum_{x=1}^{D} p(x)C_d(k-1-x,m)[F(S_{k-1})-F(S_k)]$$

$$- \sum_{k=1}^{m-1} [1-F(S_k)]c_d(1)$$

$$- \sum_{k=1}^{m-1} \sum_{x=1}^{D} p(x)C_d(k-1-x,m-1)[F(S_{k-1}) - F(S_k)]$$

$$= [1-F(S_m)]c_d(1) + \sum_{x=1}^{D} p(x)C_d(m-1-x,m)[F(S_{m-1}) - F(S_m)]$$

$$+ \sum_{x=1}^{D} p(x)C_d(m-2-x,m)[F(S_{m-2}) - F(S_{m-1})]$$

$$- \sum_{x=1}^{D} p(x)C_d(m-2-x,m-1)[F(S_{m-2}) - F(S_{m-1})]$$

$$= [1-F(S_m)]c_d(1) + \sum_{x=1}^{D} p(x)C_d(m-1-x,m)[F(S_{m-1}) - F(S_m)]$$

which is nonnegative, and the theorem is proved for $c_d \geq 0$

$c < 0.$ $\quad \Delta C_d(m-1,m-1)+c_p = [1-F(S_m)]c_d(1) \sum_{x=1}^{D} [(m+1-x)c_p$

$$+ C_d(m-1-x,m)]p(x)[F(S_{m-1}) - F(S_m)] + c_p$$

Let $m = 1$

$\Delta C_d(0,0) + c_p = [1-F(S_1)]c_d(1) + c_p$

This is positive because $c_d(1) = c_d$ and $c_d + c_p \geq 0$.

$\therefore \quad C_d(1,1) + c_p > 0$

Prove by induction. Assume $(m+1-x)c_p + C_d(m-1-x,m) > 0$

Then each term in the sum is ≥ 0 and $[1-F(S_m)]c_d(1) + c_p \geq 0$

so the RHS is ≥ 0, and this proves that $mc_p + c_d(m,m)$ is monotone increasing.

Theorem 4. Future probability of production is monotone decreasing in the inventory level. Examine $\Delta P_f(m)$, where $\Delta P_f(m) = P_f(m+1)-P_f(m)$. If $\Delta P_f(m) \leq 0$, then the theorem is proved. First it is shown true for $m = 0$, and then induction is used to show it is true for positive values of m. First, simplify the expression for $\Delta P_f(m)$.

$$\Delta P_f(m) = P_f(m+1) - P_f(m)$$

$$= F(S_{m+1}) \sum_{x=1}^{D} p(x)P_f(m+1-x)$$

$$+ \sum_{j=1}^{m+1} [F(S_{m+1-j}) - F(S_{m+2-j})] \sum_{x=1}^{D} p(x)P_f(m+1-j-x)$$

$$- F(S_m) \sum_{x=1}^{D} p(x)P_f(m-x) - \sum_{j=1}^{m} [F(S_{m-j}) - F(S_{m-j+1})] \sum_{x=1}^{D} p(x)P_f(m-j-x)$$

$$= F(S_{m+1}) \sum_{x=1}^{D} p(x)P_f(m+1-x) - F(S_m) \sum_{x=1}^{D} p(x)P_f(m-x)$$

$$+ [F(S_m) - F(S_{m+1})] \sum_{x=1}^{D} p(x)P_f(m-x)$$

$$+ \sum_{j=2}^{m+1} [F(S_{m+1-j}) - F(S_{m-j+2})] \sum_{x=1}^{D} p(x)P_f(m+1-j-x)$$

$$- \sum_{j=1}^{m} [F(S_{m-j}) - F(S_{m-j+1})] \sum_{x=1}^{D} p(x)\, P_f(m-j-x)$$

$$= F(S_{m+1}) \sum_{x=1}^{D} p(x)\, [P_f(m+1-x) - P_f(m-x)].$$

This is negative if for all k less than m+1: $P_f(k) < P_f(k-1)$. By induction if for m=0 the above result is true, then by showing $\Delta P_f(m)$ is negative the theorem is proved.

Let m = 0.

$$\Delta P_f(0) = F(S_1) \sum_{x=1}^{D} p(x)[P_f(1-x) - P_f(-x)]$$

$$= F(S_1) \sum_{x=2}^{D} p(x)[-1-(-1)] + F(S_1)P(1)(1-P_\infty-1) = -F(S_1)\, P_\infty$$

which is negative. For m > 0, $\Delta P_f(m)$ yields a negative number because the difference of the terms in the summation is negative. The difference is negative because only values of $P_f(k)$ where $k < m+1$ are used. This proves that $\Delta P_f(m)$ is negative, therefore the theorem is proved.

APPENDIX G

FORMULA FOR DETERMINING THE PROBABILITY OF FUTURE PRODUCTION

The probability of future production is the probability that the sum of the demand is greater than the inventory level. The formula is developed by using three simple examples and then generalizing to any inventory level.

If the inventory level is negative, then production occurs with certainty because demand must be satisfied. If the inventory level is zero, then production occurs if another order occurs. This happens with probability of $(1-P\infty)$.

If the inventory level is one, production occurs if an order for more than one is received before disposal, or if an order for one unit occurs before disposal and another order occurs, or if an order occurs after disposal. This probability is

$$P_f(1) = F(S_1)p(x > 1) + F(S_1)p(1)(1-P\infty) + 1-P\infty - F(S_1)$$

$$= F(S_1) \sum_{x=1}^{D} p(x)\, P_f(1-x) + [F(S_0) - F(S_1)] \sum_{x=1}^{D} p(x)P_f(-x)$$

where $P_f(k) = 1 \quad k < 0, \qquad F(S_0) = 1-P\infty$

$P_f(0) = 1-P\infty$

Examining the general case with m units in inventory, production occurs if

1. an order of size greater than m occurs before the first disposal

2. an order of size m occurs before the first disposal and another order occurs
3. an order of size less than m occurs before disposal and enough additional demand occurs to yield a negative inventory level
4. there are k units disposed before the first order occurs after the initial order, and this reorder is for more than m-k units
5. the same case as 4, except the order is for m-k units and another order occurs
6. the same case as 4, except the reorder is for less than m-k units and enough additional demand occurs to make the inventory level negative.

These six cases yield the formula below:

$$P_f(m) = F(S_m)p(x > m) + F(S_m)p(m)(1-P_\infty) + F(S_m) \sum_{x=1}^{m-1} p(x)P_f(m-x)$$

$$+ \sum_{j=1}^{m} [F(S_{m-j}) - F(S_{m-j+1})] \sum_{x=m-j+1}^{D} p(x)$$

$$+ \sum_{j=1}^{m} [F(S_{m-j}) - F(S_{m-j+1})]p(m-j)(1-P_\infty)$$

$$+ \sum_{j=1}^{m} [F(S_{m-j}) - F(S_{m-j+1})] \sum_{x=1}^{m-j-1} p(x)P_f(m-j-x)$$

This expression can be simplified by recalling that

$P_f(0) = 1-P_\infty$ and $P_f(k) = 1$ for $k < 0$.

The formula becomes

$$P_f(m) = F(S_m) \sum_{x=1}^{D} p(x)P_f(m-x)$$

$$+ \sum_{j=1}^{m} [F(S_{m-j}) - F(S_{m-j+1})] \sum_{x=1}^{D} p(x)P_f(m-j-x).$$

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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